THE OSCULATING CIRCUMFERENCE PROBLEM

School

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Abstract

The aim of the article is to study the osculating circumference i.e. the circumference which best approximates the graph of a curve at one of its points. We will define this circumference and we will describe several methods to find it. Finally we will introduce the notions of round points and crossing points, points of the curve where the circumference has interesting properties.

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Contents

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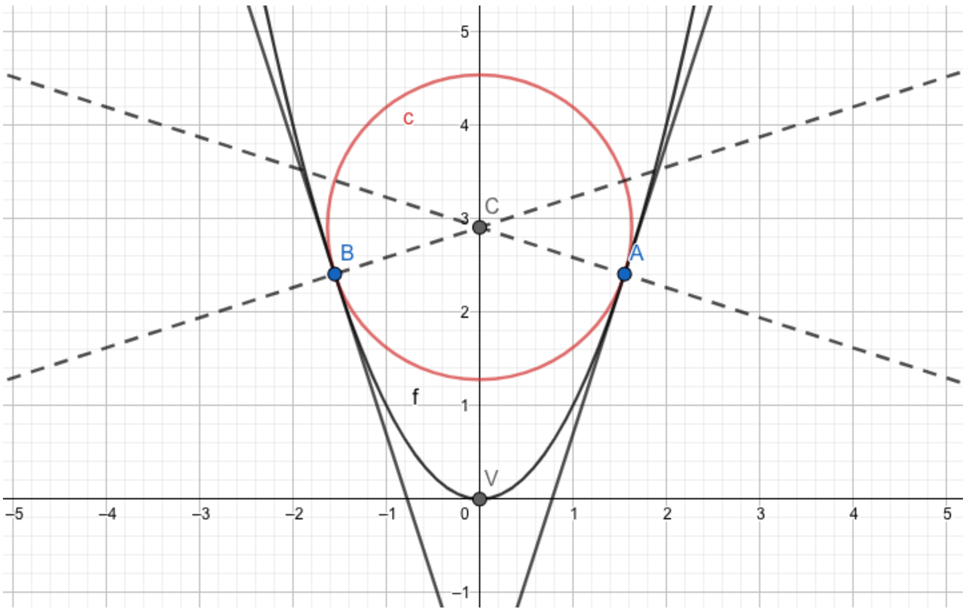
Introduction

Our research was born from an analytic geometry problem studied during the third year of high school.

THE PROBLEM. Let $\mathcal{P} : y = x^2$ be a parabola and A and B two points of the curve symmetrical about the axis of the parabola. Find the equation of the circumference $\mathcal{C}$ tangent to the parabola in the given points.
We solved the problem but it led to some interesting questions.

Let $A(x_0; x_0^2)$ and $B = (-x_0; x_0^2)$ be the two symmetrical points. We look for the center of the circumference and we notice that it must belong to the y-axis due to symmetry reasons. This center can be determined as the intersection point between the normal lines to the curve respectively at A and B.



The coordinates of the center C depend on the coordinates of the two tangency points A e B: we wondered how the coordinates of C vary when A and B vary, in particular if A and B approach to the vertix V of the parabola ($x_0 \to 0$).

Our FIRST HYPOTHESIS was that if $A \to V$ and $B \to V$ then $y_C \to +\infty$, given that the absolute value of the slopes of the normal lines increases when $A \to V$ and $B \to V$.
However we noticed that this hypothesis is not correct because there is another variation that should be considered that is the decrease of the ordinates of A and B.

Our SECOND HYPOTHESIS was that if $A \to V$ and $B \to V$ then $y_C \to y_V$. However we understood that even this guess is wrong. In fact we are able to prove that the point C approaches to a point different from V. We determine the equations of the normal lines to the curve n_A and n_B at A and B and the center C as the intersection point between the two normal lines. The equations of the tangent lines in A and B are:

$$t_A : y = 2xx_0 + x_0^2, \qquad t_B : y = -2xx_0 - x_0^2.$$

It follows that the equations of the normal lines are:

$$n_A : y = x_0^2 - \frac{1}{2x_0}(x - x_0), \qquad\qquad n_B : y = x_0^2 - \frac{1}{2x_0}(x + x_0).$$

The coordinates of C are the solutions of the system

$$\begin{cases} y = x_0^2 - \dfrac{1}{2x_0}(x - x_0) \\ \\ y = x_0^2 + \dfrac{1}{2x_0}(x + x_0) \end{cases},$$

and we get

$$C\left(0; x_0^2 + \frac{1}{2}\right).$$

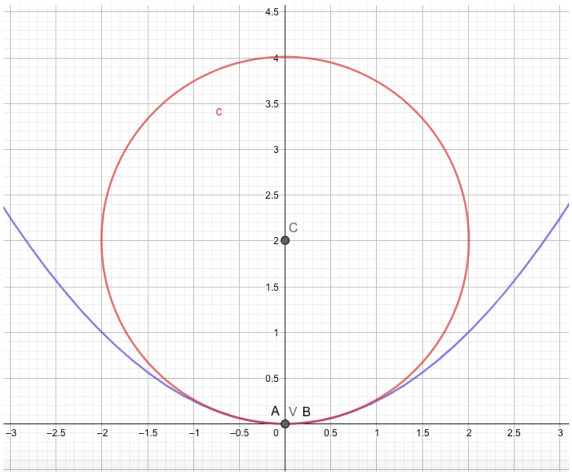
Then if $A \to V$ and $B \to V$ then $y_C \to \frac{1}{2}$. This is a consequence of

$$\lim_{A \to V} y_C = \lim_{B \to V} y_C = \lim_{x_0 \to 0} y_C = \lim_{x_0 \to 0}\left(x_0^2 + \frac{1}{2}\right) = \frac{1}{2}.$$

This means that C approaches to the point $\left(0; \frac{1}{2}\right)$. Since $\overline{CA} = \overline{CB} = \sqrt{x_0^2 + \frac{1}{4}}$ the equation of the circumference $\mathcal{C}$ tangent at the parabola at A and B is

$$\mathcal{C} : x^2 + \left(y - x_0^2 - \frac{1}{2}\right)^2 = x_0^2 + \frac{1}{4} \quad \Rightarrow \quad \mathcal{C} : x^2 + y^2 - (2x_0^2 + 1)y - x_0^4 = 0.$$

In the case in which $A \to V$ and $B \to V$ the circumference $\mathcal{C}$ approaches to the circumference of equation $x^2 + y^2 - y = 0$.



This circumference has some interesting properties:

- it intersects the parabola in the vertex V;
- it is tangent to the parabola in the vertex V;
- and finally we observed that the graphs of the circumference and of the parabola, in a neighbourhood of the vertex V, are very similar.

Here starts our research. Given a plane curve, we are now looking for the circumference that has got these three properties in one of its points.

Pieve di Soligo,
2020, June.

1 The osculating circumference

We started by giving a definition of what being "very similar" means and in order to do so we analysed the Taylor series expansion. However a Taylor series is defined for a function and we know that a circumference is not the graphic of a function so to overcome this problem we focused on the semicircumferences.

Definition 1.0.1. *Given the circumference $\mathcal{C}$ of radius r and center $C(x_C; y_C)$. The* upper semicircumference *$\mathcal{U}$ is the set of the points $(x; y) \in \mathcal{C}$ such that $y > y_C$. The* lower semicircunference *$\mathcal{L}$ is the set of the points $(x; y) \in \mathcal{C}$ such that $y < y_C$. Given $\mathcal{U}$ or $\mathcal{L}$, the circumference $\mathcal{C}$ is called the* associated circumference *to $\mathcal{U}$ or $\mathcal{L}$.*

Definition 1.0.2. *Let $f \in C^3(I)$, with $I \subseteq \mathbb{R}$, a function and a point $x_0 \in I$ such that $f''(x_0) \neq 0$. The* osculating circumference *$\mathcal{C}_f(x_0)$ of a function f in x_0 is the circumference associated to the upper semicircumference (if $f''(x_0) < 0$) or the lower semicircumference (if $f''(x_0) > 0$) which has the same Taylor series expansion to second order of f at x_0.*

Theorem 1.0.3. *Let $\mathcal{C}_f(x_0)$ be the osculating circumference of a function $f \in C^3(I)$ in $x_0 \in I$, then the radius r is given by*

$$r = \frac{\left[1 + f'(x_0)^2\right]^{\frac{3}{2}}}{\mid f''(x_0) \mid},$$

and the centre C has the following coordinates

$$C\left(x_0 - \frac{f'(x_0)}{f''(x_0)}\left[1 + f'(x_0)^2\right]; \, f(x_0) + \frac{\left[1 + f'(x_0)^2\right]}{f''(x_0)}\right).$$

Proof. Let $f \in C^3(I)$a function and $C(x_C; y_C)$ be the centre and r the radius of the osculating circumference of the function in x_0. The circumference has equation $(x - x_C)^2 + (y - y_C)^2 = r^2$. Let's distinguish four cases:

(a) $f'(x_0) > 0, f''(x_0) > 0$,

(b) $f'(x_0) > 0, f''(x_0) < 0$,

(c) $f'(x_0) < 0, f''(x_0) > 0$,

(d) $f'(x_0) < 0, f''(x_0) < 0$.

In case (a) the semicircumference has equation $y = y_C - \sqrt{r^2 - (x - x_C)^2}$ since $f''(x_0) > 0$. The second order Taylor polynomial of f at x_0 is

$$f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2,$$

while the second order Taylor polynomial of y at x_0 is

$$y(x_0) + \frac{y'(x_0)}{1!}(x - x_0) + \frac{y''(x_0)}{2!}(x - x_0)^2.$$

Given that these two expression must be equal according to the Definition 1.0.2, we have to impose that the two functions and their prime and second derivatives must be equal:

$$\begin{cases} f(x_0) = y_C - \sqrt{r^2 - (x_0 - x_C)^2} \\ \\ f'(x_0) = \dfrac{x_0 - x_C}{\sqrt{r^2 - (x_0 - x_C)^2}} \\ \\ f''(x_0) = \dfrac{r^2}{\sqrt{[r^2 - (x_0 - x_C)^2]^3}} \end{cases} . \tag{1}$$

From the second of (1) and from $f'(x_0) > 0$ we deduce that $x_0 - x_C > 0$. If we consider the second power of both sides of the second equation of (1) we get

$$(x_0 - x_C)^2 = \frac{f'(x_0)^2\, r^2}{1 + f'(x_0)^2}. \tag{2}$$

The second power of both sides of the third of (1) leads to

$$f''(x_0)^2[r^2 - (x_0 - x_C)^2]^3 = r^4; \tag{3}$$

subsituting (2) in (3) we get an explicit expression for r:

$$f''(x_0)^2 \left[r^2 - \frac{f'(x_0)^2\, r^2}{1 + f'(x_0)^2}\right]^3 = r^4 \qquad \Longrightarrow \qquad r^2 = \frac{\left[1 + f'(x_0)^2\right]^3}{f''(x_0)^2}, \tag{4}$$

and so

$$r = \frac{\left[1 + f'(x_0)^2\right]^{\frac{3}{2}}}{\mid f''(x_0) \mid}. \tag{5}$$

By substituting (4) in (2) we can find x_C:

$$x_0 - x_C = \left|\frac{f'(x_0)}{f''(x_0)}\right| \left[1 + f'(x_0)^2\right], \tag{6}$$

and so by hypothesis

$$x_C = x_0 - \frac{f'(x_0)}{f''(x_0)} \left[1 + f'(x_0)^2\right].$$

And then finally we substitute (4) and the square of (6) in the first equation of (1) in order to get y_C:

$$y_C = f(x_0) + \frac{f'(x_0)^2 + 1}{f''(x_0)},$$

where we took away the absolute value by hypothesis. By the analysis of the four cases we got the same radius (5) beacuse the signes of $f'(x_0)$ and $f''(x_0)$ do not affect the expression of r. As far the coorinadates of C are concerned we get:

(a) $C\left(x_0 - \dfrac{f'(x_0)}{f''(x_0)} \left[1 + f'(x_0)^2\right]; f(x_0) + \dfrac{\left[1 + f'(x_0)^2\right]}{f''(x_0)}\right);$

(b) $C\left(x_0 + \dfrac{f'(x_0)}{f''(x_0)} \left[1 + f'(x_0)^2\right]; f(x_0) - \dfrac{\left[1 + f'(x_0)^2\right]}{f''(x_0)}\right);$

(c) $C\left(x_0+\dfrac{f'(x_0)}{f''(x_0)}\ \left[1+f'(x_0)^2\right];f(x_0)+\dfrac{\left[1+f'(x_0)^2\right]}{f''(x_0)}\right)$;

(d) $C\left(x_0-\dfrac{f'(x_0)}{f''(x_0)}\ \left[1+f'(x_0)^2\right];f(x_0)-\dfrac{\left[1+f'(x_0)^2\right]}{f''(x_0)}\right)$.

We noticed that the combination of the signes of $f'(x_0)$ and $f''(x_0)$ lead always to the same expresssion:

$$C\left(x_0-\frac{f'(x_0)}{f''(x_0)}\ \left[1+f'(x_0)^2\right];f(x_0)+\frac{\left[1+f'(x_0)^2\right]}{f''(x_0)}\right).$$

□

Theorem 1.0.4. *Let $f\in C^3(I)$, $I\subseteq\mathbb{R}$, a function and let $x_0\in I$ be a point such that $f''(x_0)\neq 0$. Given three points $P(x_0;f(x_0))$ $Q(x_0-h;f(x_0-h))$ and $R(x_0+k;f(x_0+k))$ with $h,\ k>0$. Let C be the interception point between the normal straight lines at Q,R and $\mathcal{C}_Q$ and $\mathcal{C}_R$ be the circumferences with centre C and radius respectively equal to $\overline{QC}$ and $\overline{RC}$. If h,k approach 0, then $\mathcal{C}_Q$ and $\mathcal{C}_R$ approach $\mathcal{C}_f(x_0)$.*

Proof. Let the line $y=f(x_0-h)+m[x-(x_0-h)]$ be the tangent to the function in the point Q which slope is $m=f'(x_0-h)$. The normal line to the tangent is

$$y=f(x_0-h)-\frac{1}{f'(x_0-h)}[x-(x_0-h)].$$

In an analogous way, the normal line to the tangent at the point R is The normal line to the tangent is

$$y=f(x_0+h)-\frac{1}{f'(x_0+h)}[x-(x_0+h)].$$

Given these two equations, it is now possible to calculate the coordinates of the point C. From

$$f(x_0-h)-\frac{x-(x_0-h)}{f'(x_0-h)}=f(x_0+h)-\frac{x-(x_0+h)}{f'(x_0+h)}$$

we deduce that

$$\begin{aligned}
x&=\frac{[f(x_0+h)-f(x_0-h)]\,f'(x_0-h)f'(x_0+h)}{f'(x_0-h)-f'(x_0+h)}+\frac{(x_0+h)f'(x_0-h)-(x_0-h)f'(x_0+h)}{f'(x_0-h)-f'(x_0+h)}\\
&=\frac{[f(x_0+h)-f(x_0-h)]\,f'(x_0-h)f'(x_0+h)}{f'(x_0-h)-f'(x_0+h)}+x_0+\frac{f'(x_0-h)+f'(x_0+h)}{f'(x_0-h)-f'(x_0+h)}\\
&=\frac{[f(x_0+h)-f(x_0-h)]\,f'(x_0-h)f'(x_0+h)+h\,[f(x_0+h)+f(x_0-h)]}{f'(x_0-h)-f'(x_0+h)}+x_0.
\end{aligned}$$

By Lagrange's theorem there exists $c\in(x_0-h;x_0+h)$ such that $[f(x_0+h)-f(x_0-h)]=2h\,f'(c)$ and there exists $d\in(x_0-h;x_0+h)$ such that $f'(x_0-h)-f'(x_0+h)=-2h\,f''(d)$, then

$$x=x_0+\frac{2hf'(c)\,f'(x_0)^2+2h\,f'(x_0)}{2h\,f''(d)}.$$

As $c, d \to x_0$ the abscissa of the centre becomes

$$x = x_0 + \frac{|f'(x_0)|\,[1+f'(x_0)^2]}{|f''(x_0)|};$$

substituting the abscissa of C in the equation of the normal line, the ordinate of the centre is

$$y = f(x_0) - \frac{1+f'(x_0)^2}{|f''(x_0)|}.$$

The lenght of the radius $\overline{PC}$ is:

$$\overline{PC} = \sqrt{\frac{[1+f'(x_0)^2]^2 f'(x_0)^2}{f''(x_0)^2} + \frac{[1+f'(x_0)^2]}{f''(x_0)^2}} = \frac{[1+f'(x_0)^2]^{\frac{3}{2}}}{|f''(x_0)|}.$$

□

Theorem 1.0.5. *Let $f \in C^3(I)$, $I \subseteq \mathbb{R}$, a function and let $x_0 \in I$ be a point such that $f''(x_0) \neq 0$. Given three points $P(x_0; f(x_0))$, $Q(x_0-h; f(x_0-h))$ and $R(x_0+k; f(x_0+k))$ with h, $k > 0$. Let $\mathcal{C}_{h,k}$ be the circumference through P, Q, R; if h, k approach 0 then $\mathcal{C}_{h,k}$ approaches $\mathcal{C}_f(x_0)$.*

Proof. Let $\mathcal{C}$ be the circumference of equation $x^2+y^2+ax+by+c=0$ and radius $r = \sqrt{\left(-\frac{a}{2}\right)^2 + \left(-\frac{b}{2}\right)^2 - c}$. The circumference determinated by three points $P(x_0; f(x_0)), Q, R$ has radius equal to:

$$r(x_0,h) = \frac{\alpha(x_0,h)\,\beta(x_0,h)\,\gamma(x_0,h)}{2|h\,[2f(x_0) - f(x_0-h) - f(x_0+h)]|},$$

where

$$\alpha(x_0,h) = \sqrt{[f(x_0) - f(x_0+h)]^2 + h^2},$$
$$\beta(x_0,h) = \sqrt{[f(x_0) - f(x_0-h)]^2 + h^2},$$
$$\gamma(x_0,h) = \sqrt{[f(x_0-h) - f(x_0+h)]^2 + 4h^2}.$$

Given that $f \in C^3$, then it satisfies the Lagrange's Theorem in $[x_0-h, x_0+h]$. If we consider the interval $[x_0, x_0+h]$ then there exists a point of abscissa $i \in]x_0, x_0+h[$ such that

$$f'(i) = \frac{f(x_0+h) - f(x_0)}{h},$$

which can be expressed as $f(x_0) - f(x_0+h) = -h\,f'(i)$ and because of this α can be written as

$$\alpha(x_0,h) = \sqrt{[f(x_0) - f(x_0+h)]^2 + h^2} = \sqrt{h^2\,f'(i)^2 + h^2} = h\sqrt{f'(i)^2+1}.$$

In an analogous way, there exists a point of abscissa $j \in]x_0-h, x_0[$ such that

$$f'(j) = \frac{f(x_0) - f(x_0-h)}{h}.$$

Then $f(x_0) - f(x_0-h) = h\,f'(j)$, so

$$\beta(x_0,h) = \sqrt{[f(x_0) - f(x_0-h)]^2 + h^2} = \sqrt{h^2 f'(j)^2 + h^2} = h\sqrt{f'(j)^2+1}.$$

Similarly to the previous case, considering the interval $[x_0-h, x_0+h]$ it exists a point of abscissa $l \in]x_0-h, x_0+h[$ such that

$$f'(l) = \frac{f(x_0+h)-f(x_0-h)}{2h};$$

then $f(x_0-h)-f(x_0+h) = -2h\, f'(l)$ and so

$$\gamma(x_0,h) = \sqrt{[f(x_0-h)-f(x_0+h)]^2+4h^2} = \sqrt{4h^2 f'(k)^2+4h^2} = 2h\sqrt{f'(k)^2+1}.$$

Now it is possible to express again the radius as

$$\frac{1}{2|h\,[2f(x_0)-f(x_0-h)-f(x_0+h)]|} = \frac{1}{2|h\,[f(x_0)-f(x_0-h)-(f(x_0+h)-f(x_0))]|};$$

applying Lagrange's theorem to the radius formula it is possible to write

$$\frac{1}{2|h\,[2f(x_0)-f(x_0-h)-f(x_0+h)]|} = \frac{1}{2|h^2\,[f'(j)-f'(i)]|}.$$

Let's calculate the limit of the radius for $h \to 0$:

$$\begin{aligned}\lim_{h\to 0} r(x_0,h) &= \lim_{h\to 0}\frac{2h^3}{2|h^2\,[f'(j)-f'(i)]|}\ \sqrt{f'(i)^2+1}\ \sqrt{f'(j)^2+1}\ \sqrt{f'(k)^2+1}\\ &= \lim_{h\to 0}\frac{h^3}{|h^2[f'(j)-f'(i)]|}\ \sqrt{f'(x_0)^2+1}\ \sqrt{f'(x_0)^2+1}\ \sqrt{f'(x_0)^2+1}.\end{aligned}$$

Once again it is possible to apply Lagrange's theorem to the function in $I = [i,j]$, where $i > j$ because the point i belongs to the right neighborhood of x_0 and j to its left one. Then it exists a point of abscissa $t \in]i,j[$ such that

$$f''(t) = \frac{f'(i)-f'(j)}{i-j} \Longrightarrow f'(j)-f'(i) = -(i-j)\, f''(t).$$

Then:

$$\lim_{h\to 0} r(x_0,h) = \lim_{h\to 0}\frac{h^3}{|h^2\ [h\ f''(x_0)]|}\ \ \sqrt{f'(x_0)^2+1}\ \sqrt{f'(x_0)^2+1}\ \sqrt{f'(x_0)^2+1}$$

$$= \lim_{h\to 0}\frac{h^3\ \sqrt{(f'(x_0)^2+1)^3}}{h^3|f''(x_0)|} = \frac{[1+f'(x_0)^2]^{3/2}}{|f''(x_0)|}.$$

This last expression is equal to the expression of the radius of the osculating circle of f evaluated at $x = x_0$. □

1.1 Particular cases: the formulas cannot be applied

Once that the osculating circumference was defined we moved to look for a method to define the osculating circumference in some particular points where the formulas cannot be applied. Let's start by analysing the case in which the *left derivative does not exist.*

Example 1.1.1. *Let's consider the function $f(x) = \sqrt{x}$, in the point $O(0;0)$. Let's define a point $A(x; f(x))$ with $x \geq 0$ and let's calculate the radius of the osculating circumference in this point and make the point A approach O, using Theorem 1.0.3. In order to do this it is necessary to calculate the prime and second derivatives of the function $f(x) = \sqrt{x}$. Then*

$$f'(x) = \frac{1}{2\sqrt{x}}; \qquad f''(x) = -\frac{1}{4\sqrt{x^3}}.$$

Hence the radius is the limit of the formula previously found as x approaches 0.

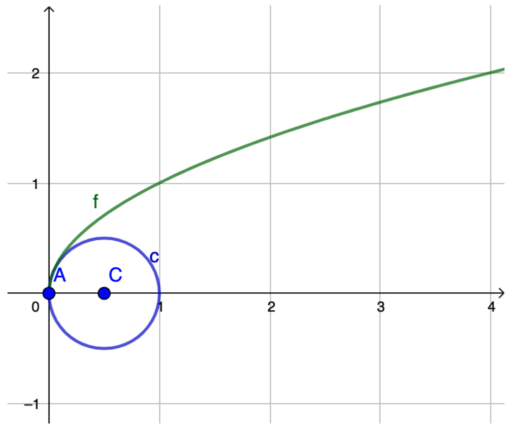
$$r = \lim_{x\to 0^+} \frac{\left[1 + f'(x)^2\right]^{3/2}}{|\, f''(x) \,|} = \lim_{x\to 0^+} \frac{\left[1 + \left(\frac{1}{2\sqrt{x}}\right)^2\right]^{3/2}}{\left|-\frac{1}{4\sqrt{x^3}}\right|} = \frac{1}{2}.$$

Thanks to Theorem 1.0.3, it is now possible to calculate the coordinates of the centre C of the osculating circumference.

$$x_C = \lim_{x\to 0^+} \left(x - \frac{f'(x)}{f''(x)} \left[1 + f'(x)^2\right]\right) = \lim_{x\to 0^+} \left(x + 2x + \frac{1}{2}\right) = \frac{1}{2};$$

$$y_C = \lim_{x\to 0^+} \left(f(x) + \frac{\left[1 + f'(x)^2\right]}{f''(x)}\right) = \lim_{x\to 0^+} \left(\sqrt{x} - (4x+1)\sqrt{x}\right) = 0.$$

Hence, the equation of the osculating circumference is $\left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$. ■



Now let's analyse the case in which the function has an *inflection point where the tangent line is vertical.*

Example 1.1.2. *Let's consider the function $f(x) = \sqrt[3]{x}$ in the point $O(0;0)$. Let's define a point $A(x; f(x))$ and $B(-x; f(-x))$ and let's calculate the radius of the osculating circumference in this point and make the points A and B approach respectively O^+ from the right side and O^- from the left side, using Theorem 1.0.3. The prime and the second derivatives are*

$$f'(x) = \frac{1}{3\sqrt[3]{x^2}}; \qquad f''(x) = -\frac{2}{9\sqrt[3]{x^5}}.$$

Hence the radius is the limit of the formula previously found as x approaches 0.

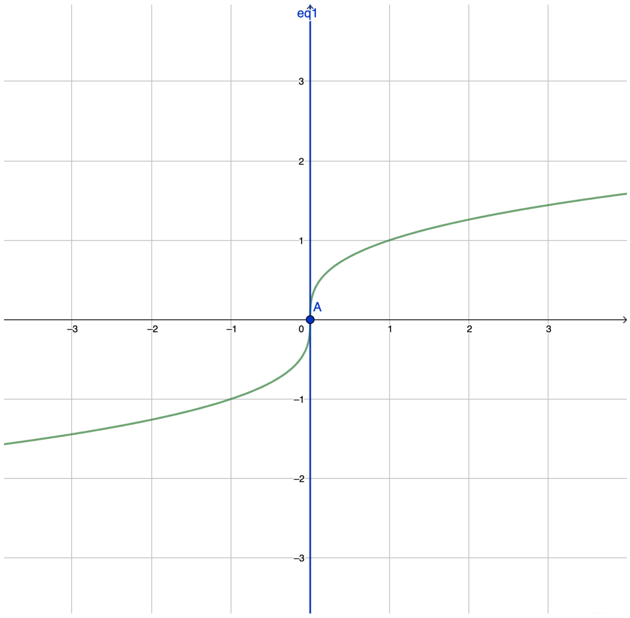
$$r = \lim_{x\to 0^\pm} \frac{\left(1 + \frac{1}{9\sqrt[3]{x^4}}\right)^{3/2}}{\left|-\frac{2}{9\sqrt[3]{x^5}}\right|} = \sqrt{\frac{\left(9\sqrt[3]{x^4} + 1\right)^3}{36\sqrt[3]{x^2}}} = +\infty.$$

Now, thanks to Theorem 1.0.3, it is possible to calculate the coordinates of the centre C of the osculating circumference:

$$x_C = \lim_{x\to 0^\pm}\left(x + \frac{3}{2}x + \frac{x}{6\sqrt[3]{4}}\right) = \pm\infty;$$

$$y_C = \lim_{x\to 0^\pm}\left[\sqrt[3]{x} + \frac{\left(1+\frac{1}{9\sqrt[3]{x^4}}\right)}{\left(-\frac{2}{9\sqrt[3]{x^5}}\right)}\right] = 0.$$

So the osculating circumference has as length of the radius $+\infty$, this means that the osculating circumference approaches a line passing through the point $O(0,0)$. Therefore, the osculating circumference degenerates into the straight line of equation $x=0$. ■



Now let's analyse the case in which the function is a *cusp*.

Example 1.1.3. *Let's consider the function $f(x)=\sqrt[3]{x^2}$ in the point $O(0;0)$. Let's define a point $A(x;f(x))$ and $B(-x;f(-x))$ and let's calculate the radius of the osculating circumference in this point and make the points A and B approach respectively O^+ and O^-, using Theroem 1.0.3, as done previously. The prime and the second derivatives are*

$$f'(x) = \frac{2}{3\sqrt[3]{x}}; \qquad f''(x) = -\frac{2}{9\sqrt[3]{x^4}}.$$

And the radius is the limit of the formula previously used as x approaches 0.

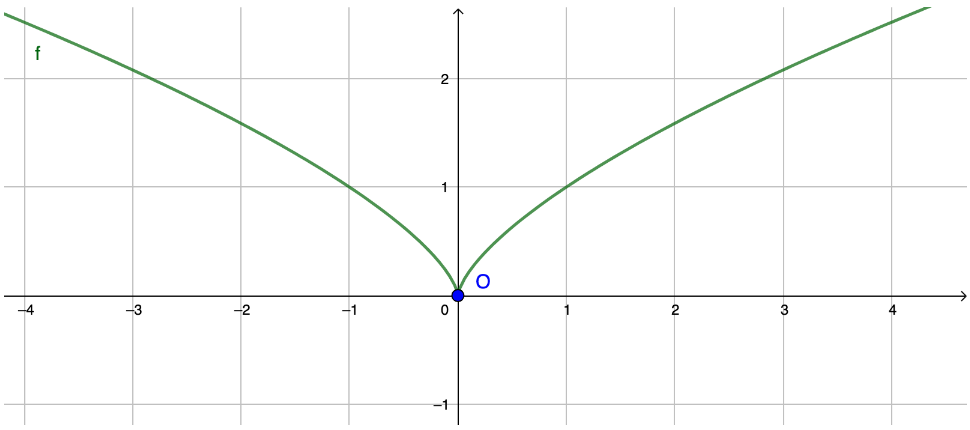
$$r = \lim_{x\to 0^\pm}\frac{\left(1+\frac{4}{9\sqrt[3]{x^2}}\right)^{3/2}}{\left|-\frac{2}{9\sqrt[3]{x^4}}\right|} = 0^+.$$

Then, it is possible to calculate the coordinates of the centre C of the osculating circumference:

$$x_C = \lim_{x\to 0^\pm}\left(x + 3x + \frac{4}{3}\sqrt[3]{x}\right) = 0^\pm;$$

$$y_C = \lim_{x\to 0^\pm}\left(\sqrt[3]{x^2} - \frac{9}{2}\sqrt[3]{x^4} - 2\sqrt[3]{x^2}\right) = 0^+.$$

These results mean that as the point A and B approaches O there are two osculating circumferences which degenerate into the point O. ■



Now let's analyse the case in which the function has a *corner point*.

Example 1.1.4. *Let's consider the function $f(x) = |x^2 - 1|$ which has two corner points $A(1;0)$ and $B(-1;0)$. Let's analyse only the point A.*
Let's define two points $D(h; f(h))$ and $E(k; f(k))$ with $h > 1$ and $0 < k < 1$. As done before we calculate the prime and the second derivatives of $f(x)$.

- *If $x > 1$ then the function is $f(x) = x^2 - 1$ and the prime and second derivatives are $f'(x) = 2x$ and $f''(x) = 2$;*
- *if $x < 1$ then the function is $f(x) = 1 - x^2$ and the prime and second derivatives are $f'(x) = -2x$ and $f''(x) = -2$.*

And now let's calculate the two radius of the osculating circumferences in the two points D and E and make them approach A, using Theorem 1.0.3, with $x_0 = 1$.

$$r_1 = \lim_{h \to 1^+} \frac{(1+4h^2)^{3/2}}{|2|} = \frac{5\sqrt{5}}{2}, \qquad r_2 = \lim_{k \to 1^-} \frac{(1+4x^2)^{3/2}}{|-2|} = \frac{5\sqrt{5}}{2}.$$

Then we calculate the coordinates of the centre of the osculating cirumference using Theorem 1.0.3; note that there are two possibilities.
For $h \to 1^+$ we get

$$x_{C_1} = \lim_{h \to 1^+} \left(h - \frac{2h}{2} \cdot (1+4h^2) \right) = -4,$$
$$y_{C_1} = \lim_{h \to 1^+} \left(h^2 - 1 + \frac{1+4h^2}{2} \right) = \frac{5}{2};$$

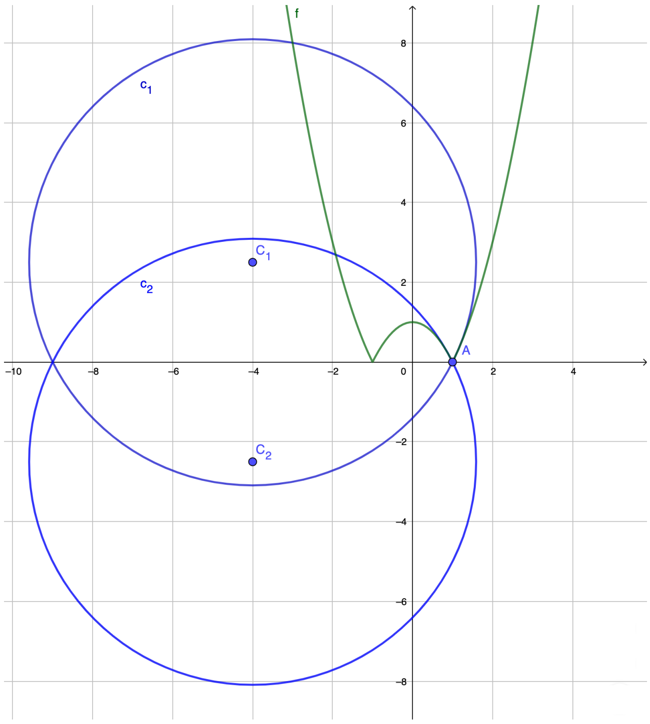
and for $k \to 1^-$ we obtain

$$x_{C_2} = \lim_{k \to 1^-} \left(k - \frac{-2k}{-2} \cdot (1+4k^2) \right) = -4,$$
$$y_{C_2} = \lim_{k \to 1^-} \left(1 - k^2 - \frac{1+4k^2}{2} \right) = -\frac{5}{2}.$$

Then there are two osculating circumferences whose equations are

$$\mathcal{C}_1 : (x+4)^2 + \left(y - \frac{5}{2} \right)^2 = \frac{125}{4} \qquad \mathcal{C}_2 : (x+4)^2 + \left(y + \frac{5}{2} \right)^2 = \frac{125}{4}.$$

■



Now let's analyse the case in which the function has an *inflection point*.

Example 1.1.5. *Let's consider the function $f(x) = x^3 - x$ in the point $O(0;0)$. Let's define two points $A(x; f(x))$ and $B(-x; f(-x))$ and let's calculate the radius of the osculating circumference in $O(0;0)$ and make the points A and B approach respectively O^+ from the right side and O^-, using Theorem 1.0.3. The prime and the second derivatives are*

$$f'(x) = 3x^2 - 1; \qquad f''(x) = 6x.$$

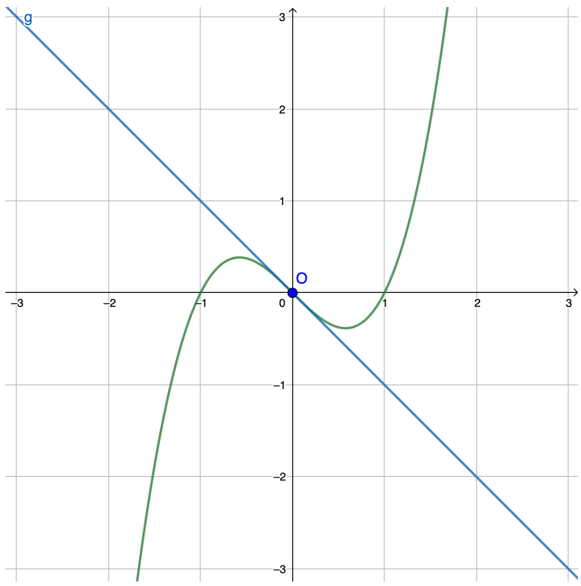
Hence the radius is the limit of the formula previously found as x approaches 0.

$$r = \lim_{x\to 0^\pm} \frac{\sqrt[3]{(9x^4 - 6x^2 + 2)^2}}{|6x|} = \frac{\sqrt[3]{4}}{|6x|} = +\infty.$$

Then it is possible to calculate the coordinates of the centre C of the osculating circumference:

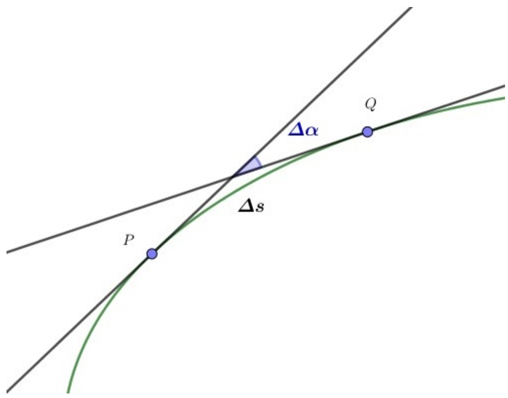
$$x_C = \lim_{x\to 0^\pm} = \frac{-27x^6 + 27x^4 - 12x^2 + 2}{6x} = \frac{1}{3x} = \pm\infty,$$
$$y_C = \lim_{x\to 0^\pm} = \frac{15x^4 - 12x^2 + 2}{6x} = \frac{1}{3x} = \pm\infty.$$

These results mean that as the points A and B approach O the osculating circumference degenerates into the straight line $y = -x$. ∎



1.2 Curvature

There is an alternative method that can lead to the osculating circumference. It starts from the simple observation that there is a direct proportion between the radius of the osculating circumference and the tendency of the curve to be like a straight line. This property suggests to analyse the variation of the angle α made by the tangent line to the graphic of the function and the x-axis.



Let $y = f(x)$ be a function derivable in an interval I and $P(x; f(x))$ and $Q(x+\Delta x; f(x+\Delta x))$ two points with $x \in I$ and Δx the increment of the variable x. Let $\Delta\alpha$ be the angle made by the two tangents lines to the graphic to the function in P and Q, then $\frac{\Delta\alpha}{\Delta s}$ is the mean variation of the angle α and so

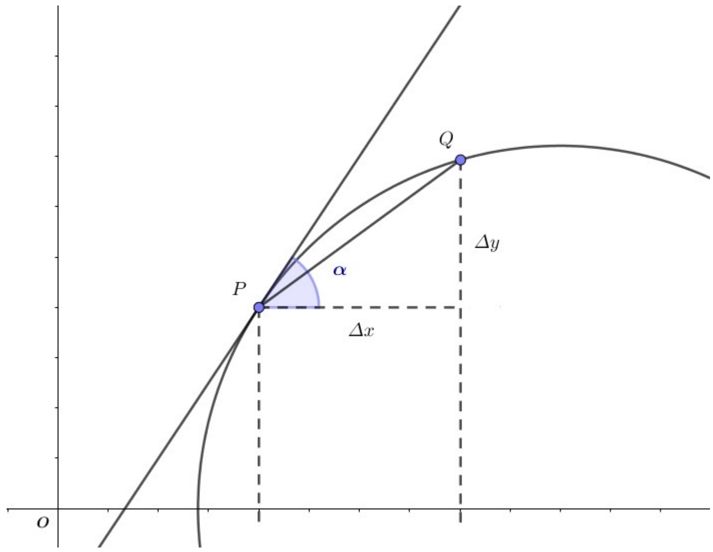
$$\lim_{\Delta x \to 0} \frac{\Delta\alpha}{\Delta s} = \frac{\mathrm{d}\alpha}{\mathrm{d}s}$$

is the variation of the angle α in the point P. We observed that the radius of the osculating circumference decreases as this limit increases so the radius is the reciprocal of the previous limit.

Therefore we calculate this limit. We know that $\tan\alpha$ is the prime derivative of $f(x)$ that is $\alpha = \arctan f'(x)$. Then we derive the composed function $\alpha(x) = \arctan f'(x)$ with respect to x and we obtain

$$\mathrm{d}\alpha = \frac{f''(x)}{1 + f'(x)^2}\,\mathrm{d}x.$$

Let's consider the two point P and Q.



We call Δs the arc length between P and Q. By Pythagoras' Theorem we have that

$$(\overline{PQ})^2 = \Delta x^2 + f(\Delta x)^2.$$

Then by dividing each member by Δx^2 we obtain

$$\left(\frac{\overline{PQ}}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2,$$

which is equal to

$$\left(\frac{\overline{PQ}}{\Delta s}\right)^2 \left(\frac{\Delta s}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2.$$

By definition, the tangent line to the graphic of the functon in the point P is the limit of the secant line that intersects the graphic in the points P and Q when Q approaches P. We calculate the limits of both members for $\left(\frac{\overline{PQ}}{\Delta s}\right)^2$ approach to 1 and we obtain

$$\left(\frac{\mathrm{d}s}{\mathrm{d}x}\right)^2 = 1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2.$$

Then we have that $\mathrm{d}s = \pm\sqrt{1 + f'(x)^2}\,\mathrm{d}x$. We finally get to the final equation of the radius of the osculating circumference:

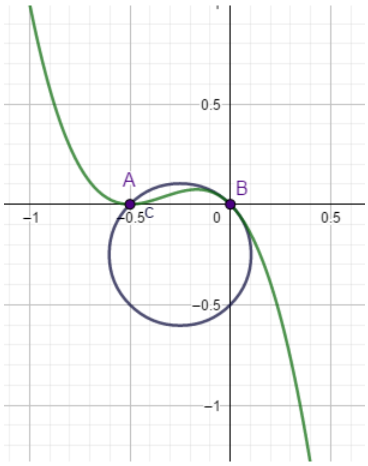
$$r = \frac{\mathrm{d}s}{\mathrm{d}\alpha} = \pm\frac{[1 + f'(x)^2]^{\frac{3}{2}}}{f''(x)},$$

which is exactly the same result obtained with the other methods.

2 Round points

From Theorem 1.0.4 we can deduce that the intersection point P between the graphic of a function f and its osculating circumference in P can be considered as three overlapping points. Let's show an example.

Example 2.0.1. *We consider the function of equation $y = f(x) = -4x^3 - 4x^2 - x$ and the point $B(0;0)$ of its graphic.*



The figure shows the graphic of the function and its osculating circumference in B. In order to find the coordinates of the intersection points we build a system between the equation of the function and the equation of the osculating circumference:

$$\begin{cases} -4x^3 - 4x^2 - x = y \\ x^2 + y^2 + \dfrac{1}{2}x + \dfrac{1}{2}y = 0 \end{cases}.$$

Solving the system, we get the equation $x^3(2x+1)(4x^2+6x+3) = 0$ which has six solutions: $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = -\frac{1}{2}$ and two complex solutions. Then the intersection points are $A\left(-\frac{1}{2};0\right)$ and $B(0;0)$ and we can observe that the point B can be considered as three overlapping points. ■

Now we want to formalize this concept.

Definition 2.0.1. *Let $p(x)$ be a polynomial in $\mathbb{R}[x]$ and a one of its roots,* the multiplicity of the root a of the polynomial $p(x)$ *is the maximum positive integer m such that $p(x)$ can be divided by $(x-a)^m$. We will call a an m-ple root.*

Definition 2.0.2. *Let $p(x)$ be a polynomial in $\mathbb{R}[x]$, $p(x) = 0$ its algebraic associated equation and a a solution of $p(x) = 0$,* the multiplicity of the solution a of the algebraic equation $p(x) = 0$ *is the multiplicity of a as a root of the polynomial $p(x)$. We will call a an m-*ple solution.

Definition 2.0.3. *Let $p(x,y) = 0$ and $q(x,y) = 0$ be two algebraic curves with $p(x,y)$ and $q(x,y)$ in $\mathbb{R}^2[x,y]$ and one common point P,* the multiplicity of intersection between the two curves at the point P *is the multiplicity of the abscissa or the ordinate of point P as solutions of the solving equation of the system of equations $p(x,y) = 0$ and $q(x,y) = 0$. We will call m-ple the intersection between the two curves or we will call P an m-ple intersection point for the curves.*

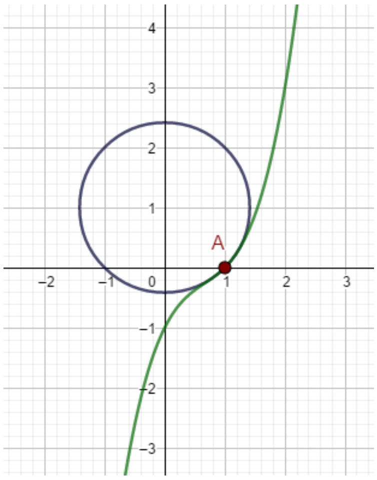
Since every function can be locally expressed by its Taylor polynomial we can extend our definitions and theorems to non-algebraic curves and curves that are not functions. Then we write a new definition suitable for our problem.

Definition 2.0.4. *Given a plane curve, one of its points P and the osculating circumference of the curve in this point, the* multiplicity of intersection between the curve and its osculating circumference in P *is defined as the multiplicity of intersection between the circumference and Taylor polynomial of the curve.*

Then, in Example 2.0.1, B is a third interception point between the cubic and the osculating circumference, because its abscissa is a third solution of the solving equation of the system made by the equations of the curves.

Let's show another example.

Example 2.0.2. *We consider now the function of equation* $y = f(x) = x^3 - 2x^2 + 2x - 1$.



In this example we can see that the two curves have only one common point A*, which is a quadruple intersection point between the two curves. In fact we get the following system of equations:*

$$\begin{cases} y = x^3 - 2x^2 + 2x - 1 \\ x^2 + y^2 - 2y - 1 = 0 \end{cases}.$$

and the solving equation $(x-1)^4(x^2+2) = 0$*, which has four solutions* $x_1 = x_2 = x_3 = x_4 = 1$*. We call* $A(1;0)$ *a* quadruple *intersection point between the curves.* ■

We can state that all the common points between the osculating circumference and the algebraic curve are at least triple points, but some are quadruple or even more.

So we have some questions: do all curves have quadruple points? Is there a way to find them? Is it possible to consider graphics of functions and not only algebraic curve?

While solving Example 2.0.1, we observed that at point $B(0;0)$ it was

- for the cubic

$$f(0) = 0, \qquad f'(0) = -1, \qquad f''(0) = -8, \qquad f'''(0) = -24;$$

- for the osculating circumference

$$\mathcal{C}_f(0) = 0, \qquad \mathcal{C}'_f(0) = -1, \qquad \mathcal{C}''_f(0) = -8, \qquad \mathcal{C}'''_f(0) = -96.$$

In Example 2.0.2 we observed that at point $A(1;0)$ it was

- for the cubic

$$f(1) = 0, \qquad f'(1) = 1, \qquad f''(1) = 2, \qquad f'''(1) = 6, \qquad f^{(4)}(1) = 0;$$

- for the osculating circumference

$$\mathcal{C}_f(1) = 0, \qquad \mathcal{C}'_f(1) = 1, \qquad \mathcal{C}''_f(1) = 2, \qquad \mathcal{C}'''_f(1) = 6, \qquad \mathcal{C}^{(4)}_f(1) = 36.$$

Therefore there is a link between the multiplicity of an intersection between the graphics of two algebraic curves at one point and their derivatives.

Theorem 2.0.5. *Let $f, g \in C^n([a;b])$ be two functions. Given $x_0 \in [a,b]$ such that $f(x_0) = g(x_0)$, if $f^{(k)}(x_0) = g^{(k)}(x_0)$, for all $k = 1, \dots, n-1$ and $f^{(n)}(x_0) \neq g^{(n)}(x_0)$ with $n \in \mathbb{N}$, then $P(x_0, f(x_0))$ is n-ple intersection point for the curves $y - f(x) = 0$ and $y - g(x) = 0$ and vice versa.*

Proof. Let's consider the two Taylor polynomial of the functions to n-th order in the neighbourhood of x_0. There exist $\xi_1 \in (x_0, x)$ and $\xi_2 \in (x_0, x)$ such that

$$\begin{cases} f(x) = f(x_0) + f'(x_0)(x-x_0) + \dfrac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \dfrac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \dfrac{f^{(n+1)}(\xi_1)}{(n+1)!}(x-x_0)^{n+1} \\ \\ g(x) = g(x_0) + g'(x_0)(x-x_0) + \dfrac{g''(x_0)}{2!}(x-x_0)^2 + \dots + \dfrac{g^{(n)}(x_0)}{n!}(x-x_0)^n + \dfrac{g^{(n+1)}(\xi_2)}{(n+1)!}(x-x_0)^{n+1} \end{cases}.$$

By putting the two conditions together and knowing that that $f(x_0) = g(x_0)$, $f^{(k)}(x_0) = g^{(k)}(x_0)$ for $k = 1, \dots, n-1$ and $f^{(n)}(x_0) \neq g^{(n)}(x_0)$ we get

$$\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(\xi_1)}{(n+1)!}(x-x_0)^{n+1} - \frac{g^{(n)}(x_0)}{n!}(x-x_0)^n - \frac{g^{(n+1)}(\xi_2)}{(n+1)!}(x-x_0)^{n+1} = 0$$

and factorizing a term $(x-x_0)^n$ we have

$$(x-x_0)^n \left[\frac{f^{(n)}(x_0)}{n!} - \frac{g^{(n)}(x_0)}{n!} + \frac{f^{(n+1)}(\xi_1)}{(n+1)!}(x-x_0) - \frac{g^{(n+1)}(\xi_2)}{(n+1)!}(x-x_0) \right] = 0.$$

Therefore, if x_0 is an n-ple solution of the equation then $P(x_0; f(x_0))$ is n-ple intersection point between the curves $y - f(x) = 0$ and $y - g(x) = 0$.

Vice versa if x_0 is an n-ple solution of the solving equation of the system above, then the polynomial

$$\begin{aligned} &f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(\xi_1)}{(n+1)!}(x-x_0)^{n+1} + \\ &- g(x_0) - g'(x_0)(x-x_0) - \frac{g''(x_0)}{2!}(x-x_0)^2 - \dots - \frac{g^{(n)}(x_0)}{n!}(x-x_0)^n - \frac{g^{(n+1)}(\xi_2)}{(n+1)!}(x-x_0)^{n+1} \end{aligned}$$

is divided by $(x-x_0)^n$ that is the polynomial

$$f(x_0) - g(x_0) + [f'(x_0) - g'(x_0)](x-x_0) + \dots + \frac{f^{(n)}(x_0) - g^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(\xi_1) - g^{(n+1)}(\xi_2)}{(n+1)!}(x-x_0)$$

is divided by $(x-x_0)^n$ therefore

$$f(x_0) = g(x_0), \qquad f'(x_0) = g'(x_0), \qquad \dots \qquad f^{(n-1)}(x_0) = g^{(n-1)}(x_0),$$

but $f^n(x_0) \neq g^n(x_0)$. □

We are going to focus on a particular type of points.

Definition 2.0.6. *Given a plane curve a* round point *is its quadruple intersection point P between the curve and the osculating circumference at P.*

Now we are able to state a helpful result in order to look for round points.

Theorem 2.0.7. *Let $f \in C^n([a;b])$ be a function, let $P(x_0; f(x_0))$ be a point of the graphic of f and let $\mathcal{C}_f(x_0)$ be the osculating circumference of f in x_0 then*

$$\mathcal{C}_f'''(x_0) = \frac{3f'(x_0)\, f''(x_0)^2}{1+f'(x_0)^2}, \tag{7}$$

$$\mathcal{C}_f^{(4)}(x_0) = \frac{3\,[1+5f'(x_0)]\; f''(x_0)^3}{[1+f'(x_0)^2]^2}. \tag{8}$$

Proof. Let's prove (7). Given $\mathcal{C} : (x-x_C)^2+(y-y_C)^2 = r^2$ the generic equationof the osculating circumference of centre $(x_C; y_C)$, we first isolate y:

$$y(x) = y_C \pm \sqrt{r^2-(x-x_C)^2}.$$

We calculate the derivatives up to the third order at x_0:

$$\begin{aligned} y'(x_0) &= \pm\frac{x_0-x_C}{\sqrt{r^2-(x_0-x_C)^2}}, \\ y''(x_0) &= \pm\frac{r^2}{[r^2-(x_0-x_C)^2]^{3/2}}, \\ y'''(x_0) &= \pm\frac{3r^2(x_0-x_C)}{[r^2-(x_0-x_C)^2]^{5/2}}. \end{aligned}$$

According to Theorem 1.0.3 the abscissa of the centre of the osculating circumference is:

$$x_C = x_0 - \frac{1+f'(x_0)^2}{f''(x_0)}\; f'(x_0) \quad \Rightarrow \quad x_0 - x_C = \frac{1+f'(x_0)^2}{f''(x_0)}\; f'(x_0).$$

So we have

$$y'''(x_0) = \pm\frac{3r^2(x_0-x_C)}{[r^2-(x_0-x_C)^2]^{5/2}} = \pm\frac{3r^2\,\dfrac{1+f'(x_0)^2}{f''(x_0)}\; f'(x_0)}{\left[r^2-\left(\dfrac{1+f'(x_0)^2}{f''(x_0)}\; f'(x_0)\right)^2\right]^{5/2}}.$$

Then by Theorem 1.0.3 the radius of the osculating circumference is given by

$$r = \frac{\left[1+f'(x_0)^2\right]^{3/2}}{f''(x_0)};$$

it follows that

$$\begin{aligned} y'''(x_0) &= \frac{3\,\dfrac{\left[1+f'(x_0)^2\right]^3}{f''(x_0)^2}\;\dfrac{1+f'(x_0)^2}{f''(x_0)} f'(x_0)}{\left(\dfrac{1+f'(x_0)^2}{f''(x_0)}\right)^5} \\ &= 3\,\frac{\left[1+f'(x_0)^2\right]^3}{f''(x_0)^2}\;\frac{f'(x_0)\,\left[1+f'(x_0)^2\right]}{f''(x_0)}\;\frac{f''(x_0)^5}{\left[1+f'(x_0)^2\right]^5} \\ &= \frac{3f'(x_0)\, f''(x_0)^2}{1+f'(x_0)^2}. \end{aligned}$$

Since that if $(x_0; y_0)$ is a round point then $y'''(x_0) = f'''(x_0)$ we get

$$f'''(x_0) = \frac{3f'(x_0)\, f''(x_0)^2}{1 + f'(x_0)^2}.$$

Let's prove now (8). The fourth derivative of the osculating circumference at x_0 is

$$y^{(4)}(x_0) = \pm \frac{3r^2 \left[r^2 + 4(x_0 - x_C)^2\right]}{[r^2 - (x_0 - x_C)^2]^{7/2}}.$$

Thanks to Theorem 1.0.3 we can say that

$$\mathcal{C}_f^{(4)}(x_0) = \frac{3f''(x_0)^3 \left[1 + 5f'(x_0)^2\right]}{[1 + f'(x_0)^2]^2}.$$

□

The previous Theorem gives us a way to find the possible round point of a function or curve. In fact in order to find a round point we have to look for the point for which is true (7) but not (8). Let's show some examples.

Example 2.0.3. *Given the function $y = f(x) = e^x$, we know that $f'(x) = f''(x) = f'''(x) = f^{(4)}(x) = e^x$ and so by substituting in (7)*

$$e^x = \frac{3e^x\, e^{2x}}{1 + e^{2x}} \quad \Rightarrow \quad e^{2x} = \frac{1}{2} \quad \Rightarrow \quad x = \ln\left(\frac{\sqrt{2}}{2}\right),$$

so the point is $P\left(\ln\left(\frac{\sqrt{2}}{2}\right); \frac{\sqrt{2}}{2}\right)$. Now we have to verify if for P the relation $\mathcal{C}_f^{(4)}(x_P) \neq f^{(4)}(x_P)$ holds. In fact, if $y = f(x) = e^x$,

$$f^{(4)}\left(\ln\left(\frac{\sqrt{2}}{2}\right)\right) = \frac{\sqrt{2}}{2}.$$

We notice that all the derivatives are equal to $\frac{\sqrt{2}}{2}$ for $x_P = \ln\left(\frac{\sqrt{2}}{2}\right)$, then

$$\mathcal{C}_f^{(4)}(x_P) = -\frac{3f''(x_P)^3 \left[1 + 5f'(x_P)^2\right]}{[1 + f'(x_P)^2]^2} = -\frac{7\sqrt{2}}{6}$$

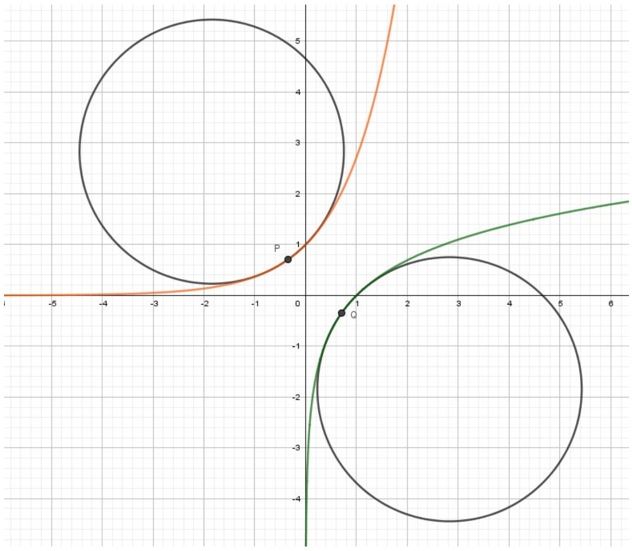
and so P is the only round point for the function $f(x) = e^x$.
Similarly, if we want to determine a round point for the function $y = g(x) \ln x$, we first calculate the derivatives of the function:

$$g'(x) = \frac{1}{x}, \qquad g''(x) = -\frac{1}{x^2}, \qquad g'''(x) = \frac{2}{x^3},$$

then we solve the equation (7):

$$\frac{2}{x^3} = \frac{3\,\dfrac{1}{x}\,\dfrac{1}{x}}{1 + \dfrac{1}{x^2}} \quad \Rightarrow \quad x = \frac{\sqrt{2}}{2}.$$

So the function $y = g(x) = \ln x$ admits only one round point of coordinates $Q\left(\frac{\sqrt{2}}{2}; \ln\left(\frac{\sqrt{2}}{2}\right)\right)$. ■



Example 2.0.4. *Given the function $y = f(x) = \sin x$, we know that $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$ and $f^{(4)}(x) = \sin x$ and so, by substituing in* (7)*,*

$$-\cos x = \frac{3\cos x\, \sin^2 x}{1+\cos^2 x} \quad \Rightarrow \quad \cos x\left(1+\cos^2 x + 3\sin^2 x\right) = 0 \quad \Rightarrow \quad x = \frac{\pi}{2} + k\pi, k \in \mathbb{Z},$$

so we obtain two sequences of such points:

$$P_k = \left\{ \left(\frac{\pi}{2} + 2k\pi; 1\right) \,\middle|\, k \in \mathbb{Z} \right\}, \qquad Q_k = \left\{ \left(\frac{3\pi}{2} + 2k\pi; -1\right) \,\middle|\, k \in \mathbb{Z} \right\}.$$

It's easy to to verify that for P_k or Q_k equation (8) *is verified. So P_k and Q_k are the round points of the function $y = f(x) = \sin x$.* ■

Example 2.0.5. *The curve of equation*[1]

$$y = \frac{1}{1+x^2}$$

admits three round points. By setting the equation defined by (7) *and solving the equation (with a numerical method) we have the following solutions*

$$x_1 \simeq -1,116424931, \qquad x_2 = 0, \qquad x_3 \simeq 1,116424931.$$

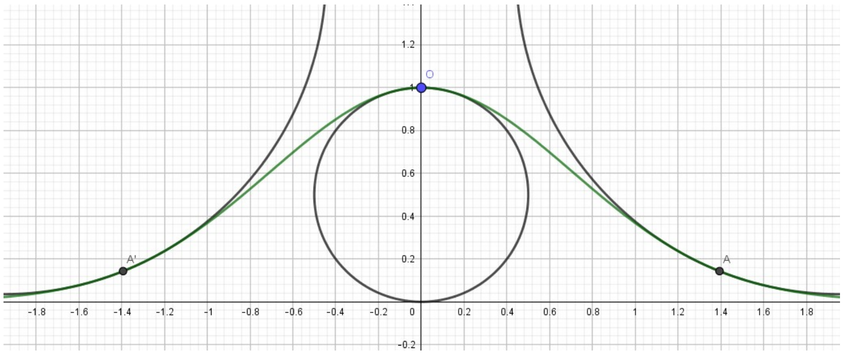
It is also possible to show that these three points are round, as their abscissa does not verify (8)*.* ■

Example 2.0.6. *The normalized Gaussian equation $y = e^{-x^2}$ admits three round points. By setting the equation defined by* (7) *and solving the equation (with a numerical method) we have the following solutions*

$$x_1 \simeq -1,393716221, \qquad x_2 = 0, \qquad x_3 \simeq 1,393716221.$$

It is possibile to show that these three points are round, as their abscissa does not verify (8)*.* ■

[1] This is a particular case of the so called *witch of Agnesi*.

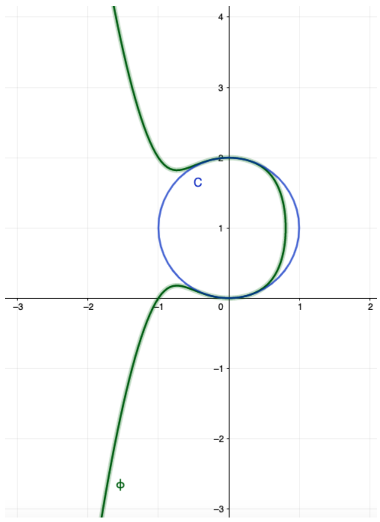


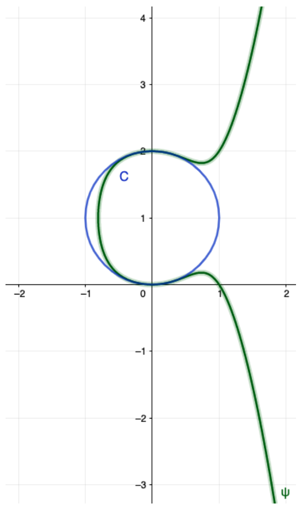
3 Beyond round points

We wondered if there exist curves which admit m-uple points with $m > 4$. We found this

Example 3.0.1. *Let's consider the two following curves:*

$$\phi(x,y) : x^2 + y^2 - 2y + x^5 = 0, \qquad \psi(x,y) : x^2 + y^2 - 2y - x^5 = 0.$$





It's easy to verify that $\mathcal{C}_\phi, \mathcal{C}_\psi : x^2 + y^2 - 2y = 0$ is the osculating circumference of ϕ and ψ in the origin. By implicit differentiation, for all $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$:

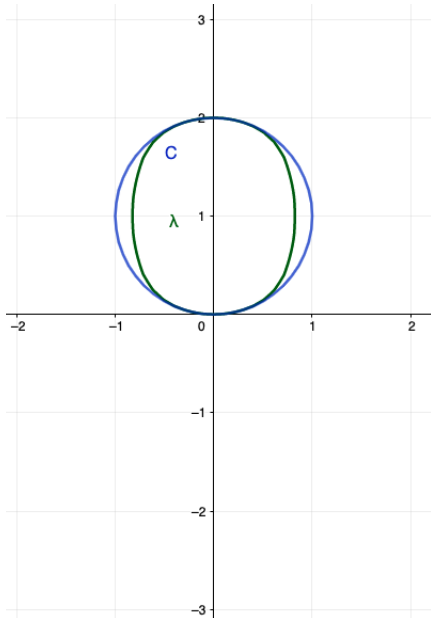
$$D^n[\phi(x,y)] = D^n[\mathcal{C}_\phi(x,y)] + D^n[x^5] \quad \text{and} \quad D^n[\psi(x,y)] = D^n[\mathcal{C}_\psi(x,y)] - D^n[x^5]$$

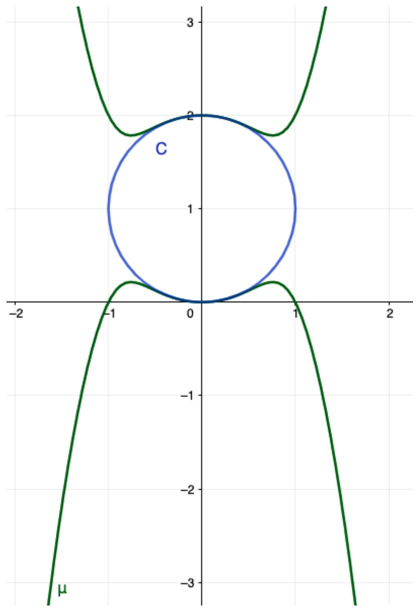
Moreover, if $x = y = 0$ then $D^n[x^5] = 0$ for $n < 5$ and $D^n[x^5] = 5! \neq 0$ for $n = 5$. So in $(0;0)$ it is true that $D^n[\phi(x,y)] = D^n[\mathcal{C}_\phi(x,y)]$ for $n < 4$ while $D^n[\phi(x,y)] \neq D^n[\mathcal{C}_\phi(x,y)]$ for $n = 5$. In conclusion the point $(0;0)$ is a quintuple intersection point for ϕ and $\mathcal{C}_\phi$. Same for $\psi, \mathcal{C}_\psi$. ■

Let's show another

Example 3.0.2. *Let's consider the two following curves:*

$$\lambda(x,y) : x^2 + y^2 - 2y + x^6 = 0, \qquad \mu(x,y) : x^2 + y^2 - 2y - x^6 = 0.$$





In an analogous way to what has been done in the previous example, we can state that the point $(0;0)$ *is a sextuple intersection point for* λ *and* $\mathcal{C}$ *and same for* μ. ■

Example 3.0.3. *Now we are able to generalize the previous results and study the pencil of curves of equation*

$$\gamma_m : x^2 + y^2 - 2y \pm x^m = 0. \tag{9}$$

We can state that $(0;0)$ *is a* m*-uple intersection point for* γ_m *and* $\mathcal{C}_{\gamma_m}$. ■

3.1 Crossing points

Let's consider again the pencil of curves (9). From the graphics of the previous examples we can observe a different behaviour of the curves in a neighbourhood of $(0;0)$ when m is even rather than when it is odd.

Definition 3.1.1. *Let* $f, g \in C^n([a;b])$ *be two functions and let* $x_0 \in [a,b]$ *be such that* $f(x_0) = g(x_0)$. *We say that* f crosses g *if* $f(x) \gtrless g(x)$ *in a left neighbourhood of* x_0 *and* $f(x) \lesseqgtr g(x)$ *in a right neighbourhood of* x_0*; we call* $(x_0; f(x_0))$ *a* crossing point *of* f *and* g.

Theorem 3.1.2. *Let* $f, g \in C^n([a;b])$ *be two functions and let* $x_0 \in [a,b]$ *be such that* $f(x_0) = g(x_0)$. *If* $f^{(k)}(x_0) = g^{(k)}(x_0)$, *for* $k = 1, 2, \ldots, n-1$ *and* $f^{(n)}(x_0) \neq g^{(n)}(x_0)$, *then if, and only if,* n *is odd, then* $(x_0; f(x_0))$ *is the crossing point of* f *and* g.

Proof. Let's suppose that $x_0 = 0$. If n is odd, we consider the two Taylor polynomials, with convenient ξ, ζ in a neighbourhood of 0, such that:

$$f(x) = f(0) + f'(0)x + \cdots + \frac{f^{(n)}(0)}{n!}x + \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1},$$
$$g(x) = g(0) + g'(0)x + \cdots + \frac{g^{(n)}(0)}{n!}x^n + \frac{g^{(n+1)}(\zeta)}{(n+1)!}x^{n+1}.$$

Let's suppose $f^{(n)}(0) > g^{(n)}(0)$. Let's calculate the following limit:

$$\begin{aligned}\lim_{x\to 0^-}[f(x)-g(x)] &= \lim_{x\to 0^-}\left[\frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1} - \frac{g^{(n)}(0)}{n!}x^n + \frac{g^{(n+1)}(\zeta)}{(n+1)!}x^{n+1}\right] \\ &= \lim_{x\to 0^-}\left(\frac{f^{(n)}(0)}{n!} - \frac{g^{(n)}(0)}{n!}\right)x^n + \lim_{x\to 0^-}\left(\frac{f^{(n+1)}(\xi)}{(n+1)!} - \frac{g^{(n+1)}(\zeta)}{(n+1)!}\right)x^{n+1};\end{aligned}$$

since x approaches 0, we can leave the second term out and we get

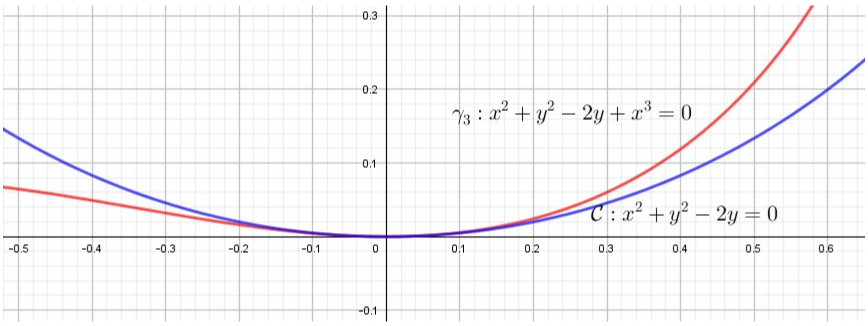
$$\lim_{x\to 0^-}\left(\frac{f^{(n)}(0)}{n!} - \frac{g^{(n)}(0)}{n!}\right)x^n = 0^-.$$

We conclude that $\lim_{x\to 0^-}[f(x)-g(x)] = 0^-$, then when x approaches 0^-, $f(x) < g(x)$.

For the same reasons $\lim_{x\to 0^+}[f(x)-g(x)] = 0^+$, then when x approaches 0^-, $f(x) > g(x)$.

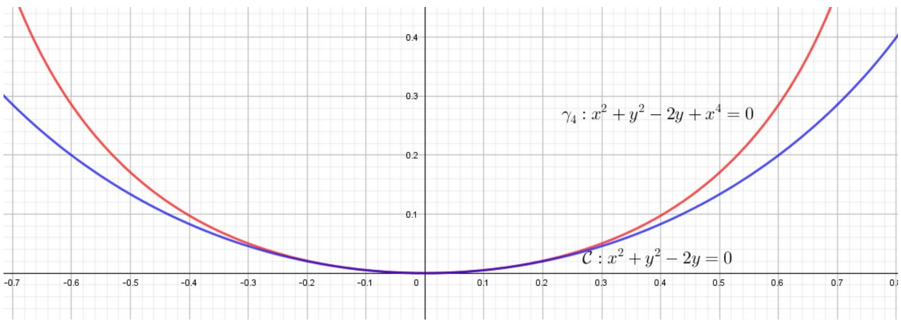
It is easy to prove that if n is even then $\lim_{x\to 0^-}[f(x)-g(x)] = 0^+$ and $\lim_{x\to 0^+}[f(x)-g(x)] = 0^+$. Therefore when x approaches 0^-, $f(x) > g(x)$ and when x approaches 0^+, $f(x) > g(x)$. This shows the other implication. □

Example 3.1.1. *In the following figure you can see the behavior of the two curves $\gamma_3 : x^2 + y^2 - 2y + x^3 = 0$ (in red) and $\mathcal{C}_{\gamma_3} 6^2 + y^2 - 2y = 0$, its osculating circumference in the origin (in blue). As you can see $\gamma_3 < \mathcal{C}_{\gamma_3}$ for $x < 0$ and $\gamma_3 > \mathcal{C}_{\gamma_3}$ for $x > 0$.*



So the origin is a crossing point of γ_3 and $\mathcal{C}_{\gamma_3}$. ■

Example 3.1.2. *In the following figure you can see the behavior of the two curves $\gamma_4 : x^2 + y^2 - 2y + x^4 = 0$ (in red) and $\mathcal{C}^2_{\gamma_4} + y^2 - 2y = 0$, its osculating circumference in the origin (in blue). As you can see $\gamma_4 > \mathcal{C}_{\gamma_4}$ for $x < 0$ and $\gamma_4 > \mathcal{C}_{\gamma_4}$ for $x > 0$.*



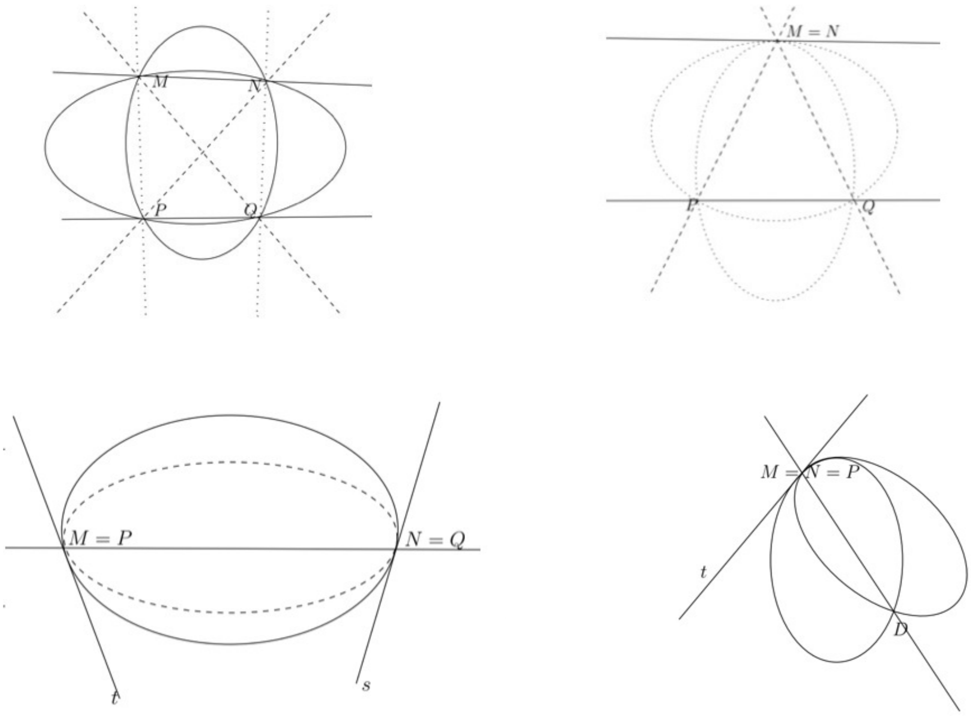
So the origin is not a crossing point of γ_4 and $\mathcal{C}_{\gamma_4}$. ■

4 Osculating circumference of a conic section

In order to find the osculating circumference of a conic section we can use the two following methods that do not involve the concept of derivative but are based on the use of Euclidean geometry properties.

4.1 The first method

The first method consists in finding the osculating circumference among the conics that belong to a pencil of conics. Two conics $\mathcal{C}$, $\mathcal{C}'$ determine a pencil of conics passing through the four common points of $\mathcal{C}$, $\mathcal{C}'$. There are four different types of pencils of conics that pass through four points:



We need to find the fourth type, that is the one in which the three points M, N and P concide so that the conics of the pencil are osculating in that point, by Theorem 1.0.5.

It is possibile to prove that the equation of this pencil is the sum between the equation of the given non-degenerate conic and the degenerate conic given by the product between the tangent to the conic in the tangency point and a straight line passing through that point. Among all the conics of this pencil we want to find the circumference therefore we impose that the coefficients of x^2 and y^2 must be equal and the coefficient of xy must be null.

Let's consider a conic whose equation is $p(x, y) = 0$, one of its points $A(x_0; y_0)$, the equation $t(x, y) = 0$ of the tangent line to the conic in A and the equation $a(x - x_0) + b(y - y_0) = 0$ of a generic straight line passing through A, then the equation of the pencil is

$$p(x, y) + t(x, y) \cdot [a(x - x_0) + b(y - y_0)] = 0.$$

This equation is equivalent to a parametric quadratic equation in x, y with a and b real numbers. By imposing the previously mentioned conditions on the coefficients of x^2, y^2 and xy we find the osculating circumference.

Example 4.1.1. *Let $\mathcal{E}$ be the ellipse of equation $5x^2 + 9y^2 - 81 = 0$ and $A(3;2)$ one of its points, we can find the equation $t : 5x+6y-27 = 0$ of the tangent line to $\mathcal{E}$ in A and we consider the pencil of lines $\gamma : a(x-3)+b(y-2) = 0$ passing through A with a, b real numbers.*
The equation of the pencil of conics Γ is

$$(5x^2 + 9y^2 - 81) + (5x + 6y - 27) \cdot [a(x-3) + b(y-2)] = 0,$$

that is

$$(5a+5)x^2 + (6b+9)y^2 + (6a+5b)xy + (-42a-10b)x + (-18a-39b)y + 81a + 54b - 81 = 0.$$

We want to find the circumference that belongs to this pencil and in order to do so we solve this system:

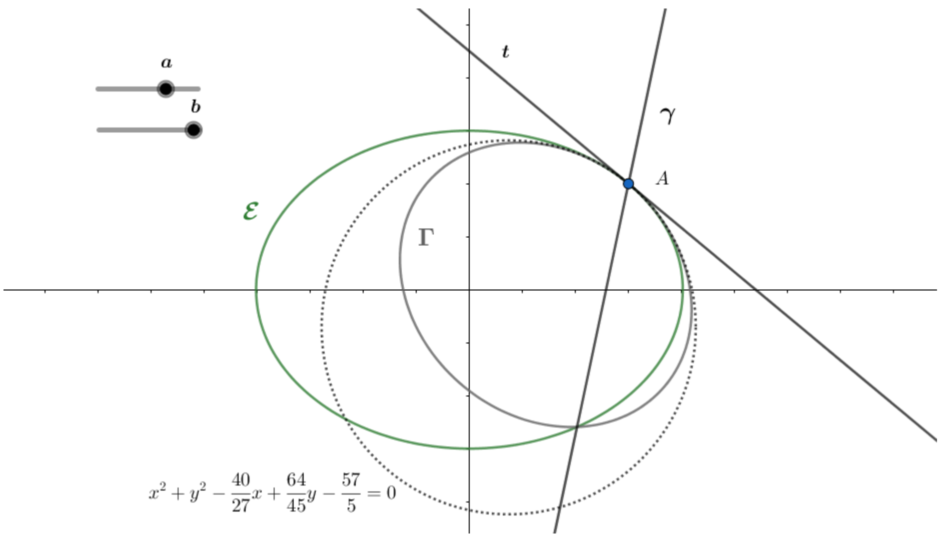
$$\begin{cases} 5a+5 = 6b+9 \\ 6a+5b = 0 \end{cases},$$

and its solution is $a = \dfrac{20}{61}$, $b = -\dfrac{24}{61}$.

Therefore, by substituting, the equation of the osculating circumference is

$$x^2 + y^2 - \frac{40}{27}x + \frac{64}{45}y - \frac{57}{5} = 0.$$

■



This method makes it possibile to find the osculating circumference without introducing the concept of derivative but it works only for the osculating circumference of a conic.

4.2 The second method

The second method is a graphic method because it lets us draw the osculating circumference without knowing its equation, through the ruler and the compass construction that is the consutruction of geometric figures using only an idealized ruler and compass as the ancient Greek mathematician used to do.

Theorem 4.2.1. *Given a non-degenerate conic $\mathcal{P}$ and one of its points P, let t be the tangent line to $\mathcal{P}$ in P. All the circumferences tangent to t in P that intersect the conic determine chord parallel to each other.*

Proof. In this proof we are going to consider the point P coincident with $O(0;0)$ and the line t such that its equation is $y = 0$ without loss of generality. Let's consider the equation of a conic:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

with $A, B, C, D, E, F \in \mathbb{R}$.
We do not consider the case $A = C \wedge B = 0$ because we obtain a circumference, which is not interesting.

By hypothesis $P(0;0)$ must belong to the conic so F must be equal to 0. Moreover the line t is tangent to the conic in the point P. Then we consider the system:

$$\begin{cases} Ax^2 + Bxy + Cy^2 + Dx + Ey = 0 \\ y = 0 \end{cases}.$$

We impose that the discriminant of the quadratic equation $Ax^2 + Dx = 0$ is equal to zero.

$$\Delta = D^2 = 0 \quad \Longrightarrow \quad D = 0;$$

therefore the equation of the conic is

$$Ax^2 + Bxy + Cy^2 + Ey = 0; \tag{10}$$

Then we intersect the conic with a circumference tangent to t in $P(0;0)$, i.e. we solve the system composed by the equation (10) and the pencil whose equation is $x^2 + y^2 + ky = 0$ with $k \in \mathbb{R}$:

$$\begin{cases} Ax^2 + Bxy + Cy^2 + Ey = 0 \\ x^2 + y^2 + ky = 0 \end{cases}.$$

The system has four $\mathbb{R}^2$-solutions at most, and two of them coinciding with P.
In order to find the other two possible solutions we solve the system by using the Gaussian elimination and we got the following system:

$$\begin{cases} Bx + (C - A)y + E - Ak = 0 \\ x^2 + y^2 + ky = 0 \end{cases}.$$

In the case $A \neq C \wedge B = 0$ we will have horizontals chords, while if $A = C \wedge B \neq 0$ we will have vertical chords, no matter what k is. In the general case $A \neq C \wedge B \neq 0$, we can write the system as

$$\begin{cases} x = \dfrac{A - C}{B}y + \dfrac{Ak - E}{B} \\ \left[\dfrac{A - C}{B}y + \dfrac{Ak - E}{B}\right]^2 + y^2 + ky = 0 \end{cases}.$$

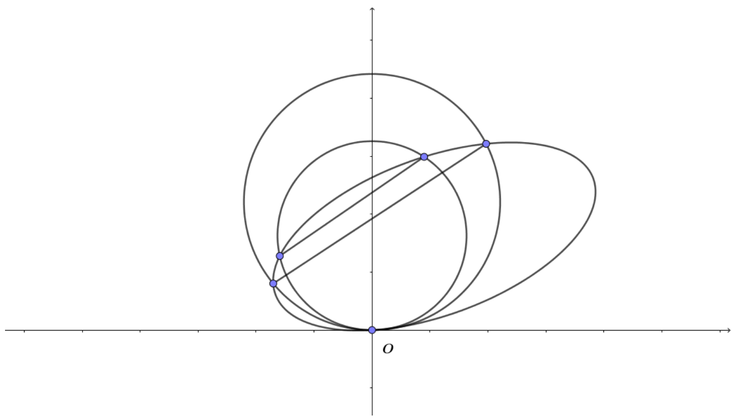
Let $I_1(x_1;y_1), I_2(x_2;y_2)$ the two distinct intersection points bewteen the conic and the circumference that do not coincide with P. Then it follows that

$$\begin{cases} y = y_1 \\ x_1 = \frac{A-C}{B}y_1 + \frac{Ak-E}{B} \end{cases} \qquad \vee \qquad \begin{cases} y = y_2 \\ x_2 = \frac{A-C}{B}y_2 + \frac{Ak-E}{B} \end{cases}.$$

We recall that the slope m of a straight line is given by $m = \frac{\Delta y}{\Delta x}$; therefore the slope of the secant line which is the infinite extension of the chord that joins I_1 and I_2 is

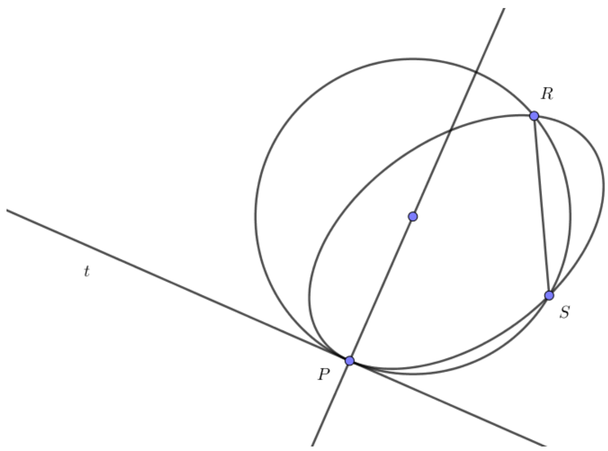
$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{B}{A - C}.$$

Since the slope of this line depends only on the coefficients of the equation of the conic, it is always the same no matter what the k is. In this way it is proved that the chords determined by the intersection points between a conics and a circumference are always parallel to each other.

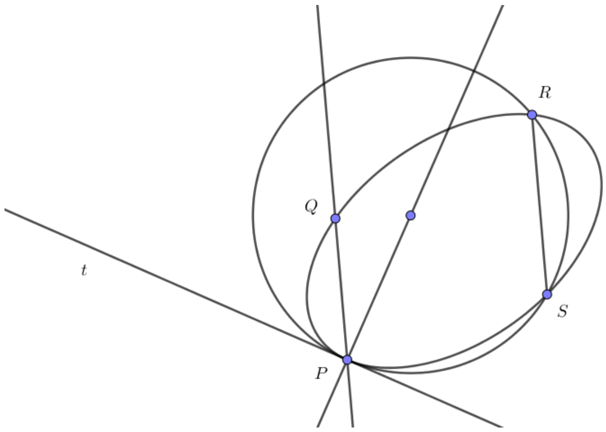


$\square$

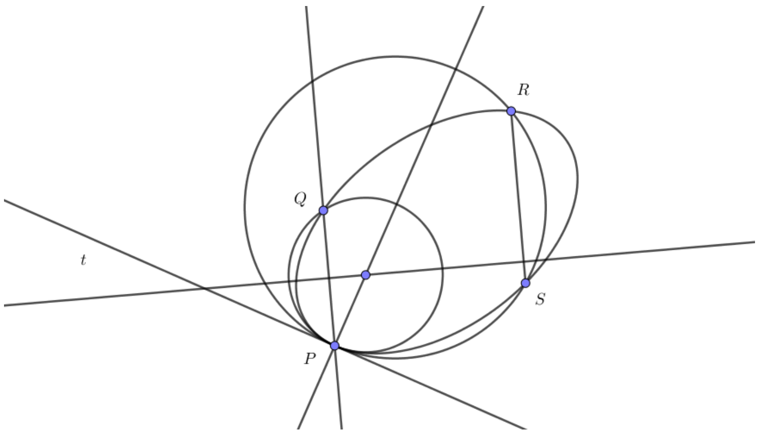
Thanks to Theorem 4.2.1 it is possibile to find the osculating circumference of all the conics in one of its points. Let's consider for instance an ellipse $\mathcal{E}$ and $P \in \mathcal{E}$. We draw the tangent line t to $\mathcal{E}$ in P and a circumference such that t is tangent to the circumference in P. The circumference intersects the ellipse in other two points R and S.



All the circumferences tangent to the line t in P intersect the ellipse in two points that make a chord: according to Theorem 4.2.1 all these chords are parallel. So we draw a line parallel to the chord RS passing through P that intersects the ellipse in a point Q.



The smaller circumference tangent to t in P intersects $\mathcal{E}$ only in Q so the three other points of intersection between the circumference and the ellipse are all coincidents to P. Therefore by Theorem **??** the circumference tangent to t in P that intersects $\mathcal{E}$ in Q is the osculating circumference of the ellipse. In order to draw the osculating circumference we draw the axis of the chord QP and the intersection point between this axis and the normal to $\mathcal{E}$ in P is the centre of the osculating circumference.



Theorem 4.2.2. *Given a non-degenerate conic $\mathcal{P}$ and one of its points P, let t be the tangent line to $\mathcal{P}$ in P and let t' be the symmetrical line to t about one of the axes of the conic*[2]*. The line t' is parallel to the chords determined by the intersection points between the conic and the circumferences tangent to t in P.*

Proof. In this proof we are going to consider the point P coincident with $O(0;0)$ and the line t such that its equation is $y = 0$, without loss of generality. As we did in the previous proof, we intersect the conic with a circumference tangent to t in $P(0;0)$ whose equation is $x^2+y^2+ky = 0$. This leads to the following system

$$\begin{cases} x = \dfrac{A-C}{B}y + \dfrac{Ak-E}{B} \\ \left[(A-C)^2 + B^2\right] y^2 + \left[2(A-C)(Ak-E) + B^2k\right] y + (Ak-E)^2 = 0 \end{cases} .$$

In the case in which the second equation has only one solution, there is just an intersection point whose coordinates are

$$P'\left(\frac{B[Ak+Ck-2E]}{2[(A-C)^2+B^2]}; -\frac{2(A-C)(Ak-E)+B^2k}{2[(A-C)^2+B^2]}\right);$$

[2]One of the two axes, in the case of an ellipse or a hyperbola, the only one in the case of a parabola.

this point is symmetrical to P about one of the axes of the conic for both the circumference and the conic, so the tangent line to the conic in P' will be symmetrical to t about the axis, and we will call this line t'.

We know that the slope of a tangent line to a conic of equation $Ax^2+Bxy+Cy^2+E=0$ in a point $(x_0;y_0)$ is given by its derivative function, that can be obtained by implicit differentiation.

$$m=-\frac{2Ax+By}{Bx+2Cy+E}.$$

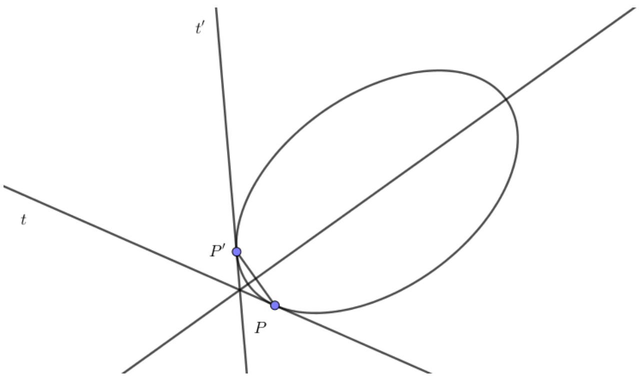
Therefore by substituting the coordinates of P' we got that the tangent line t' has slope

$$m_{t'}=\frac{B}{A-C},$$

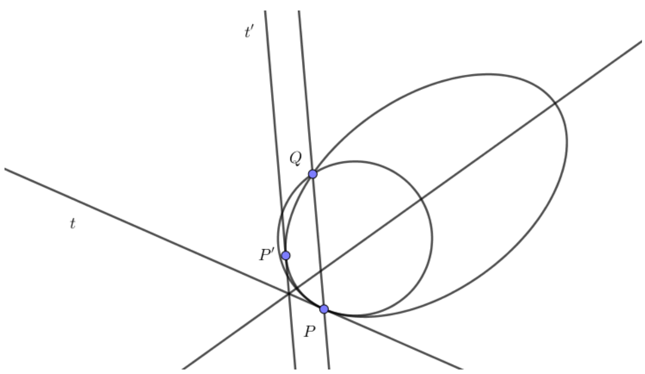
which is exactly the slope of the extension line of the chords determined by the intersection point between the conic and the osculating circumference of the conic in the point P. $\square$

It is possibile to define another version of this method that does not involve any circumference but allows us to immediately build the osculating circumference.

Let's consider for instance an ellipse $\mathcal{E}$, one of its points P and the tangent line t to $\mathcal{E}$ in P. We call P' the symmetrical point of P about one of the axes of $\mathcal{E}$ and t' the tangent line to $\mathcal{E}$ in P'.



By Theorem 4.2.2 the line t' is parallel to all the chords determined by the intersection points between the ellipse and the circumferences tangent to t in P. So we draw the parallel line to t' passing through P that intersects $\mathcal{E}$ in Q and we proceed exactly as in the previous case. The tangent circumference to t in P passing through Q is the osculating circumference of the ellipse in the point P.



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