

# AVALANCHE SIMULATOR

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## Presentation of the research topic: AVALANCHE SIMULATOR

*Lets consider a  $n \times n$  grid with an integer number of snow flakes. If in a moment a cell contains at least 4 flakes, it is unstable: it avalanches giving all its flakes to the 4 neighbouring cells. If the avalanching cell is on the border, the flakes that would fall outside the grid are considered lost*

### **Brief presentation of the conjectures and results obtained:**

We analyzed the evolution of the system according to the imposed rules with reference to the proposed problem, starting from some restrictive hypothesis. We put all  $N$  flakes in the central cell of an unlimited grid  $n \times n$ , without counting any lost flake. Flakes propagate until reaching stability after a number of steps, occupying a certain number of cells that let determinate a maximum radius  $r$  that is the maximum number of cells neighbouring to the central one. In this work we will analyze the relation between the number  $N$  of flakes in the starting cell and the radius  $r$  of the stability's configuration.

# Contents

Formulation of the problem and restrictive hypotesis . . . . . 3  
Simulation tools . . . . . 3  
Data analysis . . . . . 4  
Conclusions . . . . . 8

## Formulation of the problem

We put in each cell of a grid of  $n \times n$  cells an integer number  $N$  of snow flakes. Let this be the starting configuration. The model evolves as follows: if a cell  $z(x, y)$  where  $x$  and  $y$  are respectively line  $x$  and column  $y$ , contains at least 4 flakes, it's called unstable and loses 4 flakes which switch to the four neighbouring cells, according to the rule:

$$\begin{aligned}z(x, y) &\rightarrow z(x, y) - 4; \\z(x \pm 1, y) &\rightarrow z(x \pm 1, y) + 1; \\z(x, y \pm 1) &\rightarrow z(x, y \pm 1) + 1.\end{aligned}$$

If this cell is on the border, the flakes which should be given to the neighbouring cells are lost. In this way every unstable cell determines the evolution of the grid's state.

## Restrictive hypothesis

We fixed some restrictive hypothesis to have an easier approach to the problem:

1.  $N$  flakes were placed only in the central cell of the grid  $n \times n$ ;
2. no restriction wasn't put to the grid's dimensions, so for the moment the condition of loss of flakes was excluded.

## Simulation tools

We used a numerical approach to simulate the avalanche, building appropriate softwares that, given the starting situation (insertion of the  $N$  flakes in the central cell), simulate the evolution of the system until reaching the condition of stability, where no cell contains a number of flakes  $z > 3$ . Basically, the algorithmic approaches consist in two nested loops that produce the evolution in every cell.

## Commutativity of the algorithm

One of our first questions was the commutativity of the algorithm, that is we asked ourselves if the final state of the matrix for every step depends or not on the order of its journey. We noticed that, checking line by line or column by column, the results are the same with all the built tools.

## Spreadsheet

First we used an Excel spreadsheet to simulate the evolution of the system after putting  $N$  flakes in the central cell. The evolution was simulated with two different algorithms:

1. using the library functions of the program, building operations between cells using conditional instructions;
2. building a macro in VB script which automatically gives the configuration of the next step specifying when the system is stable. The two proceedings give the same results. The second one is more advantageous because it allows a check of every step.

## C++ algorithm

The C++ algorithm expects the initialization and the upload of the data relating to the configuration in the form of a matrix with side  $n$ . Then the program individually checks every cell and, when it finds a value greater than or equal to 4, it applies the following algorithm:

```

pos[i][j]-=soglia;

pos[i][j-1]=++;

pos[i][j+1]=++;

pos[i-1][j]=++;

pos[i+1][j]=++;

```

It goes on like this for each cell of the matrix until it completes a full step. Then it searches the maximum value  $z$  in each cell of the matrix and compares it with the limit ( $z_{max} = 4$ ). If  $z \geq z_{max}$ , it checks again the matrix applying the same algorithm for every unstable cell. The program continues until the maximum value, that is searched at the end of every step of the matrix, is less than the limit in such a way that all the cells are stable.

## Java algorithm

This java application's main purpose is the graphical representation of the propagation of the number  $N$  of flakes in the central cell through a color printing. Unlike the C++ algorithm, this program always saves the position of unstable cells, and then it works only in those, just like the algorithm in C++. Here below the main part of the algorithm is reported:

```

List<int[]> rowList = new ArrayList<int[]>();
for(int i=1;i<=bounds;i++){
for(int j=1;j<=bounds;j++){
if(mat[i][j]>=4)
rowList.add(new int[] {i,j});
}
}
for (int[] is : rowList) {
crolla(is,mat);
}

```

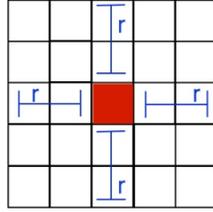
## Data analysis

### Analyzed variables

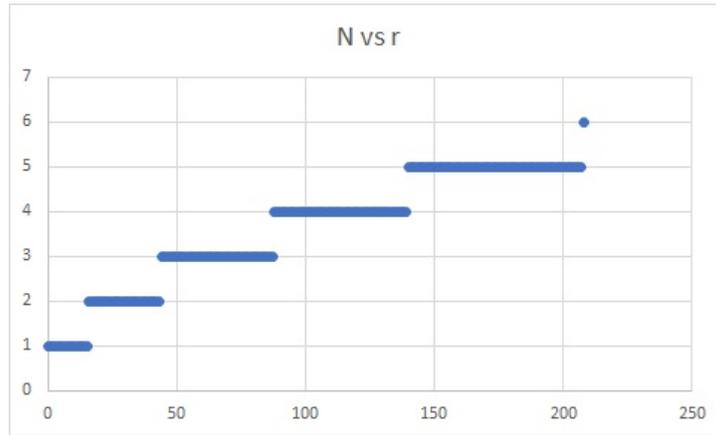
Starting from an initial configuration with  $N$  flakes in the central cell, we made the system evolve until reaching a stability situation, so that no cell contains a number of flakes  $z > 3$ . In this situation we determined the radius  $r$  defined as greatest distance in cells from the central point. As said before, we chose to study the evolution of the avalanche defining two parameters:

- the number of flakes initially contained in the central cell  $N$ ;
- the radius  $r$ , defined as the greatest distance in cells from the central one;

Beginning from a starting configuration with  $N$  flakes in the central cell, we made the system evolve until the equilibrium, where no cell contains a number of flakes  $z > 3$ . In this situation we determined the radius  $r$ .

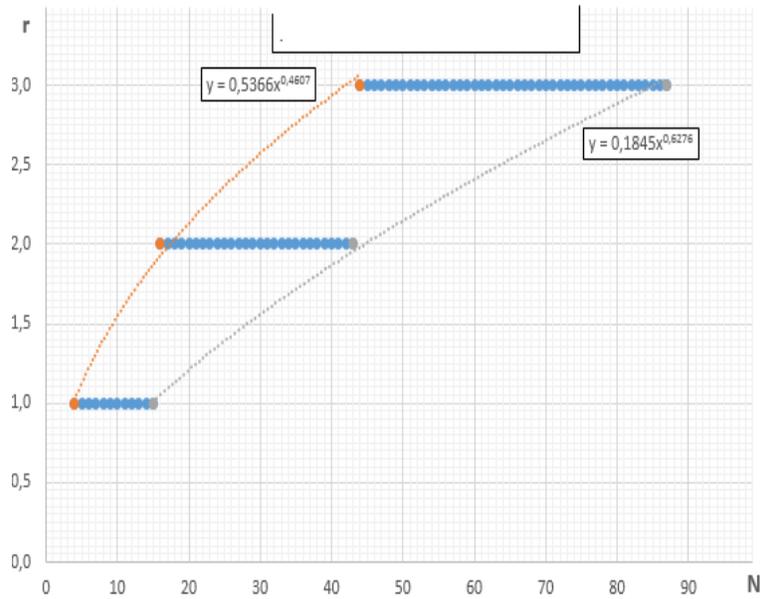


All procedures we used to simulate the avalanche unequivocally show that a direct relation between the variables  $N$  e  $r$  subsists, meaning that, as we always expect, increasing the number  $N$  of flakes in the central cell, the radius  $r$  tends to increase, too. We simulated the evolution of the system with our instruments until  $N = 20000$ .



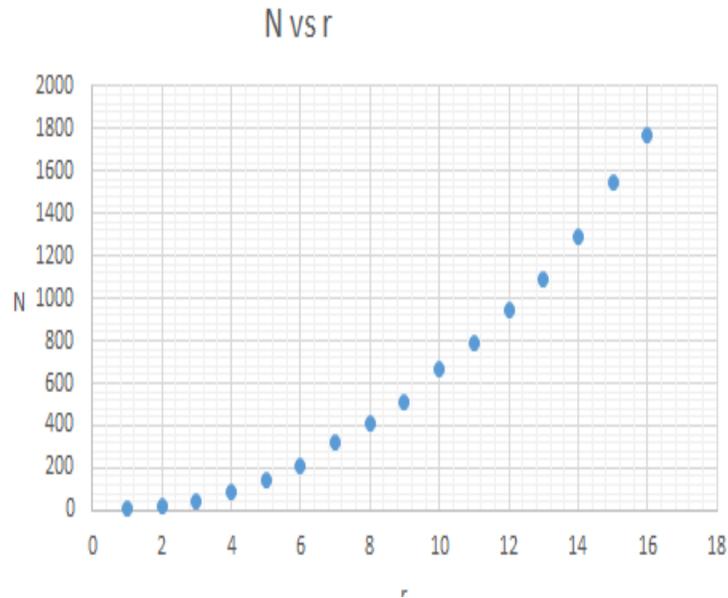
The graph above shows the intervals of  $N$  with constant radius  $r$ .

It is right to hypothesize the presence of a certain functional relation between the two variables like  $r = f(N)$ , that will be a step function. So we decided to represent the relation between the number  $N$  of the start of every step and the corresponding radius  $r$ .



The graph represents two curves that pass through two points  $P(N; r)$ , where  $N$  indicates respectively the starting and the ending of the step, defined as the range where the radius  $r$  is constant.

If you examine the inverse relation between  $N$  and  $r$  you can notice the presence of a quadratic relation like  $N \simeq r^2$ .



We fitted experimental data in a polynomial model of second degree using the automatic procedure of Excel.

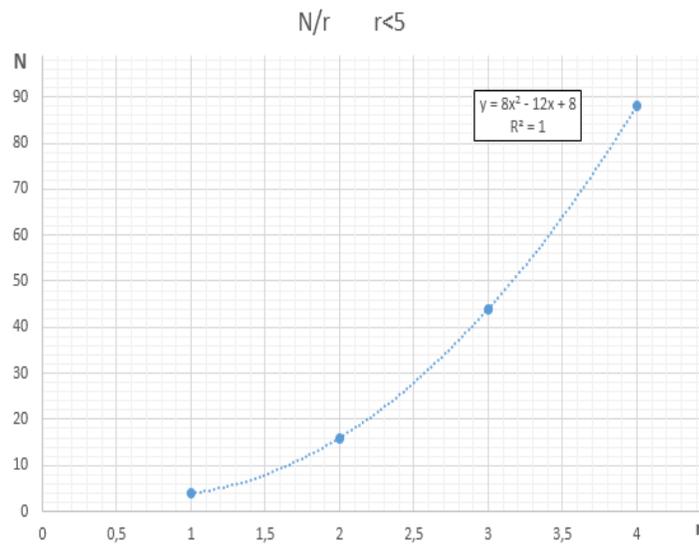
However we got an exact fit on the points relating to  $r < 5$  but not on the ones greater than  $r$ .

**$r \leq 4$  fit**

We realized a graph only with the first 4 couples of values  $r$  and  $N$ . If you have a look to the graph below, you can see that a parabola of equation:

$$N = 8r^2 - 12r + 8$$

passes through all the points  $P(r; N)$ , starting from  $r = 1$  and  $N = 4$  to  $r = 4$  and  $N = 88$ .



Note the integer values of the coefficients.

As we will discuss below, this law is not compatible with high values of  $r$ .

### $r > 4$ fit

When the number of flakes  $N$  is greater than 144 and then  $r$  is greater than or equal to 5, the quadratic function (parabola) stops fitting exactly. Indeed coefficients resulting from the fit are no longer integer numbers and have consistent variations when you add a new point.

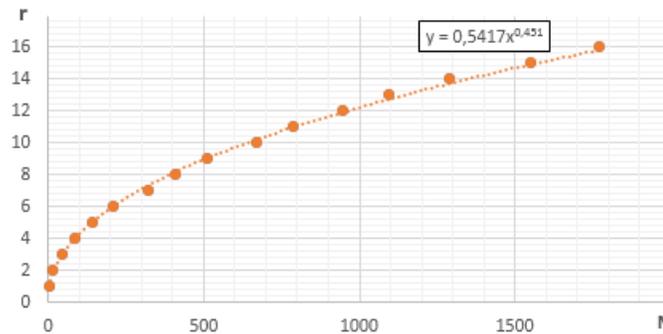
So it is necessary to represent the trend of the points with a different law. Following the example [1], we tried to search a relation that was a power law like:

$$r(N) = a \cdot N^b$$

However, we need to study the relationship  $r(N)$  and not  $N(r)$ .

We continued to determine the couple of parametres  $a$  and  $b$  through the spreadsheet's automatic function, which allows a better approximation of experimental data for all the 16 couples of available values  $N$  and  $r$ , finding out:

$$a = 0,5417, \quad b = 0,4510$$



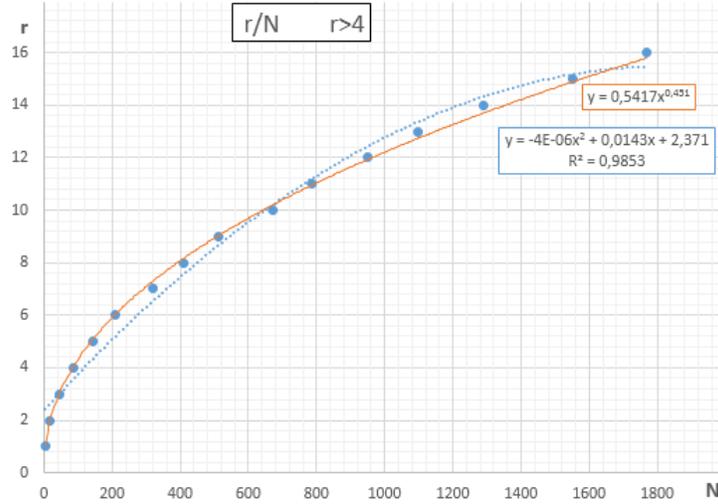
In the graphic you can see that the power law constitutes an acceptable approximation of the trend of experimental points.

### Comparison between the two models

We compared the two models: parabola and power law. For the approximation with a polynomial function of second degree we had to reverse the relationship  $N(r) = ar^2 + br + c$  writing it as:

$$r = \frac{-b + \sqrt{b^2 - 4a(c - N)}}{2a} \quad (1)$$

In order to get a criterion to establish which one of the two models (parabola and power law) best describe the phenomenon, we considered 4 couples of points, then we added a couple of points every time until  $r = 16$ , we calculated the fit's parameters and the squared deviation relative to the relations given by Excel graphs and defined as the difference between the observed value and the calculated one. Calculating the sum of the squared deviations, given that the sum of the squared deviations is equal to 0, unlike what happens in the relation given by the power law, we can notice that the parabola is more precise than the power law if we consider the first 4 points. Increasing the number of couples of points, the sum of the squared deviations calculated on the relation given by the parabola is no longer equal to 0. Even though the relation given by the power law is not precise, the sum of squared deviations is anyway less than the one given by the parabola. Here below is reported a graph representing the two curves.



We can notice from the graph that the power law's curve is much more precise than the parabola's curve, allowing a better approximation of the experimental data.

The power law is indicated by the following equation:

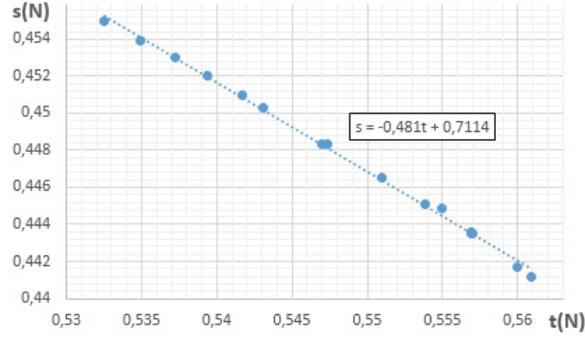
$$y = 0,5417x^{0,451}$$

As we already noticed, increasing the number of the points used in the fit, we often found consistent variations of the  $a$  and  $b$  parameters of the polynomial model, whereas the parameters of the power law model have more similar values, even though they present variations.

$r$	$N$	$a(N)$	$b(N)$
5	144	0,5569	0,4436
6	208	0,5550	0,4449
7	320	0,5609	0,4412
8	408	0,5600	0,4417
9	512	0,5570	0,4435
10	672	0,5569	0,4435
11	788	0,5538	0,4451
12	948	0,5510	0,4465
13	1096	0,5470	0,4483
14	1288	0,5473	0,4483
15	1552	0,5431	0,4503
16	1768	0,5417	0,451
17	1960	0,5394	0,452
18	2208	0,5372	0,453
19	2456	0,5349	0,4539
20	2708	0,5325	0,455

Table 1: Coefficient  $a(N)$  and exponent  $b(N)$  obtained from the power law graphic for every  $r$  and  $N$ ,  $r = a \cdot N^b$

Observing the data reported in the table [1], we wonder if a relation between  $a(N)$  and  $b(N)$  could exist. Using 20 couples of values of  $a(N)$  and  $b(N)$ , we searched for a fit between these two parameters noting the presence of a linear relation.

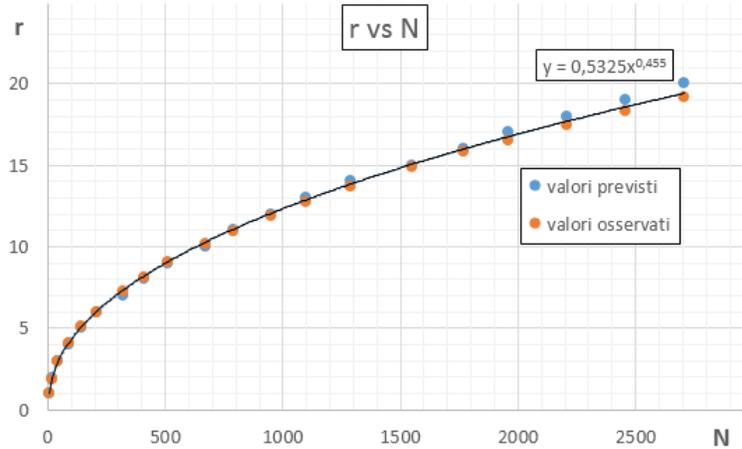


The relation that emerges is:

$$b = -0,481a + 0,7114 \quad (2)$$

From this, we understand that the power law that approximates the trend of the data depends on a single parameter. If we choose the exponent  $b(N)$  as the only parameter, the relation of the best fit is:

$$r = \left( \frac{0,7114 - b}{0,481} \right) \cdot N^b \quad (3)$$



In this graph we reported the theoretic trends calculated with the law (3) and the experimental data. Fitting the relation (3), the model that better fits the data is

$$r = 0,5325N^{0,455}$$

This trend is in accordance with [1.] where the exponent tends to  $\frac{1}{2}$  for an high  $N$ . We also noticed that for a small  $N$  the model could be approximated by the relation (1): in the limit of a high  $N$  we find a relation like:

$$r \simeq N^{\frac{1}{2}}$$

So we can understand that if we added a consistent number of points, the value of  $b$  of best fit would tend to  $\frac{1}{2}$ . According to the relation (2), if  $b = \frac{1}{2}$ , then  $a \simeq 0,44$ .

## Interpretation of the model

To understand the relation that we found between  $N$  and  $r$  we can proceed as follows.

In the starting situation we put  $N$  flakes in the central cell of the grid  $n \times n$ . According to the criteria that we defined for the evolution of the system,  $N$  flakes are distributed on the entire structure until reaching

the stability configuration without losing any of them. Then we can think that the number of all the flakes stays constant: only its spatial distribution will change, from a starting situation where the  $N$  flakes occupy the volume of the central column, to the final situation, where a certain volume corresponding to a certain number of columns is occupied.

We define the occupied volume  $V$  related to a certain cell as the volume of a squared-based parallelepiped with unitary side and height  $h$  equal to the number of flakes in the cell.

Obviously the stability situation imposes that:

$$0 \leq h \leq 3$$

In this way the starting volume of the central column is:

$$V_0 = N$$

After the evolution of the system, the final configuration, after a certain number of step  $t$ , will have a certain number of occupied cells that, as we have seen, have a circular symmetry around the central cell. The higher the starting number of flakes  $N$ , the better this symmetry is. In the graph below the final configuration obtained putting  $N = 1 \cdot 10^6$  starting flakes is registered.

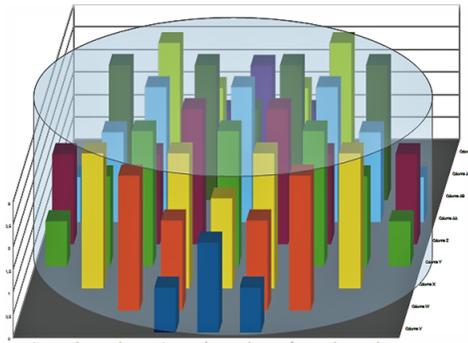


As you can see, the distribution has an evident circular symmetry. Different shades of grey are used to represent the occupation values of cells in the final state with the darkest corresponding to 3 flakes and white for 0. The radius of the figure is equal to the matrix's radius.

The total volume occupied by the flakes will be constant because the total number of flakes  $N$  must be preserved.

The distribution of the occupied cells around the central one with respect to the number  $N$  of flakes is very complex, but still allows to define the parameter  $r$ . So, as stated before, we can try to make an estimate of the total volume occupied by the flakes.

To do so, we approximate the final cells configuration with a cylinder of radius  $r$  and height  $h$ .



The cylinder's volume is:

$$V = \pi r^2 \cdot h$$

Considering the volume  $V$  equal to  $N$ , we obtain that:

$$r = \frac{\sqrt{\pi h N}}{\pi h} = a \cdot N^{\frac{1}{2}}$$

This is consistent with the experimental model we found, where  $a \simeq 0,4$ . In fact, we note that for this value of  $a$  we obtain:

$$0,4 = \frac{\sqrt{\pi h}}{\pi h} \Rightarrow h \simeq 2$$

This means that on average the height of every column included on the cylinder of radius  $r$  is high  $h = 2$ , so this means that the average number  $\bar{z}$  of flakes in every cell on the equilibrium configuration is equal to 2.

We expect that the average value is  $\bar{z} = 1,5$ , considering that the number of flakes for every balanced cell is  $0 \leq z \leq 3$ . However, our simulation show that for high values of  $N$  we would expect to find a low number of cells with  $z = 0$  of flakes according to the total number of cells included in the basic circumference of the cylinder of radius  $r$ : this could lead to rise the average number  $z$  according to 1,5, leading it to a  $\bar{z} \simeq 2 = h$ .

## Conclusions

In addressing the avalanches problem, we tried to find a mathematical model that best approximated the relation between  $r$ , radius of the stability configuration and  $N$ , the number of flakes in the central cell.

We found that for an high  $N$  the relation can be approximated with a power law like:

$$r = a \cdot N^b$$

According to our simulations, the parameter  $b$  tends to the value  $b = 0,5$ , in agreement with [1.], while the parameter  $a$  tends to the value  $a = 0,44$ , in agreement with the fact that the average number  $h$  of flakes in each cell in the final stability configuration of the avalanche is near the value  $h \simeq 2$ , noting that the whole configuration assume a circular symmetry.

We cant provide a rigorous proof of the result nevertheless were confident of it!

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